International Journal on Science and Technology (IJSAT)
E-ISSN: 2229-7677 ● Website: www.ijsat.org ● Email: editor@ijsat.org

Mathematical Analysis of Epidemic Dynamics Under Periodic Transmission Patterns

Neeraj Kumar

Department of Mathematics
Government Degree College Budaun (UP) 243601

Abstract:
Periodic changes in disease transmission are critical to understand the temporal behaviour of infectious diseases. The dynamics of an epidemic can be affected by seasons, environmental conditions, population behavior, contact patterns, and periodic social activities, and hence, the transmission rate of an epidemic can vary systematically over time. This paper presents a mathematical analysis of epidemic dynamics under periodic transmission patterns for a non-autonomous compartmental modelling framework. The transmission coefficient is represented as a positive periodic function to allow for recurrent changes in the transmission of the disease and to still keep the basic complexity of classical epidemic models. We consider the existence, positivity and boundedness of the epidemic solutions and study the behaviour of disease-free and endemic states under time-dependent transmission. In particular, we pay special attention to epidemic thresholds, local stability, disease persistence and extinction in periodically forced epidemic systems. We discuss the impact of transmission amplitude, periodicity, recovery rate and demographic parameters on the path of the epidemic. The mathematical framework suggests that one should distinguish between constant and periodically varying transmission since systems that have similar average transmission rates may have very different time-dependence. We introduce the concept of Floquet theory to understand the stability of periodic epidemic states and relate them to mathematical theories of recurrent outbreaks. We then discuss the importance of periodic transmission for vaccination, treatment and other time-dependent intervention strategies and solutions to the above problems. We find that the temporal periodicity of epidemic models provides a more flexible theoretical framework to interpret recurrent outbreaks, disease persistence, and long-term behaviour of infectious diseases.

Keywords: Periodic transmission; Mathematical epidemiology; Epidemic dynamics; Non-autonomous epidemic models; SIR model; SEIR model; Seasonal forcing; Disease persistence; Disease extinction; Epidemic threshold; Periodic solutions; Floquet theory; Infectious diseases; Transmission dynamics; Mathematical modelling.

1. Introduction

Mathematical modelling is an important tool for understanding the transmission and long-term behaviour of infectious diseases. Epidemic models provide a systematic model for describing the interaction of susceptible, infected, exposed and recovered individuals and how demographic and epidemiological parameters affect the progression of an outbreak. Classical compartmental models, particularly susceptible-infected-recovered (SIR) and susceptible-exposed-infected-recovered (SEIR) models, have

been studied in mathematical studies of infectious diseases. These models have been used to investigate epidemic thresholds, disease persistence, extinction, outbreak size and effectiveness of control measures
A fundamental assumption in most classical epidemic models is that the transmission rate is constant over time. This simplifies the mathematical analysis but does not represent the variation in time that is observed in real epidemic systems. The probability of disease transmission can change with the seasons, human behaviour, school and work-life cycles, population movement, social groupings and other patterns of contact can change regularly as well. Hence the transmission process may have a temporal structure, which is better described by the time-dependent transmission coefficient rather than a constant one. Periodic transmission introduces a key mathematical difference between autonomous and non-autonomous epidemic systems. In an autonomous model, the governing parameters are not explicitly dependent on time and a periodically forced epidemic model has coefficients satisfying the relation

$$\boldsymbol{\beta}(\boldsymbol{t}+\boldsymbol{T}) = \boldsymbol{\beta}(\boldsymbol{t}),$$

where $(\boldsymbol{\beta}(\boldsymbol{t}))$ represents the transmission rate and $(\boldsymbol{T} > \boldsymbol{0})$ is the period of the forcing. This way the mathematical model can describe recurrent changes in transmission while preserving the basic compartmental structure of the epidemic system.
The introduction of temporal periodicity can significantly affect epidemic dynamics. A disease can undergo recurrent outbreaks even when the corresponding constant-parameter model has reached an equilibrium. The amplitude and period of transmission variation can also affect the timing, magnitude, and persistence of epidemic outbreaks. So the average value of the transmission coefficient alone may not fully characterize the long-term behaviour of a periodically forced epidemic system. The timing of high- and low-transmission phases may also affect the evolution of the infected population.
From a mathematical point of view, periodically varying epidemic models naturally lead to the theory of non-autonomous ordinary differential equations. They study the existence and uniqueness of the solutions, the positivity of the epidemiological variables, the limits of the population, the stability of disease-free states, the persistence of infection and the possibility that periodic epidemic solutions exist. Unlike autonomous systems, where equilibrium points are the main objects of long-term analysis, periodically forced systems may have time-dependent periodic states. The stability of these states can be studied using tools of periodic differential equations and Floquet theory. The epidemic threshold is also one of the major aspects of periodic variations in transmission. In constant transmission models, the basic reproduction number is a common measure that can be used to determine if an infection can enter a susceptible population. When the transmission changes over time, the invasion criterion has to account for the time structure of the coefficients. The threshold then can not only depend on the average transmission intensity of the disease, but also the periodic forcing and the interaction of the transmission variation with other epidemiological parameters. Periodic epidemic models are particularly relevant for diseases with strong seasonal behaviour. Many infectious diseases show recurring increases and decreases in incidence at regular times, either due to environmental interference or human activity. A mathematical model of such phenomena could help to distinguish between transient outbreaks and persistent recurrent disease dynamics. It might also be useful to study how change in the amplitude or frequency of periodic transmission affects epidemic persistence and outbreak recurrence.
The periodic formulation is also relevant to epidemic control. Vaccination, treatment, public health interventions and behavioural changes may change in time. For example, the intensity of interventions

may be higher during high transmission and lower during lower transmission. This means understanding the mathematical response of an epidemic system to periodic forcing can provide a theoretical basis to study time-dependent intervention strategies. The goal is not just to determine whether an infection can be maintained or not, but to understand how temporal variation of epidemiological processes leads to qualitative changes in the dynamics of the underlying dynamical system. We have presented a mathematical framework for studying epidemic dynamics under periodic transmission patterns. We study qualitative properties of periodically forced compartmental epidemic models such as solution behaviour, disease-free and endemic dynamics, epidemic thresholds, stability, persistence and extinction. We pay particular attention to the effects of the amplitude and period of transmission change and the mathematical difference between constant and periodic transmission.

The paper is organized as follows. In section 2 we provide the mathematical description of an epidemic model with periodic transmission and discuss basic assumptions and properties of the model. In section 3 we investigate the qualitative dynamics of the model, such as positivity, boundedness, disease-free behaviour and epidemic thresholds. In section 4 we discuss periodic epidemic behaviour and the mathematical implications of periodic solutions and Floquet theory. In section 5 we discuss the biological interpretation and implications of periodic transmission patterns while section 6 we present extensions and future research directions. We conclude this paper by emphasising the mathematical significance of temporal periodicity of epidemic models.

2. Mathematical Formulation of Epidemic Models with Periodic Transmission

The mathematical description of an epidemic with periodically varying transmission requires a non-autonomous dynamical system in which at least one epidemiological parameter depends explicitly on time. The periodicity may represent seasonal changes, recurring social behaviour, environmental variations, or other mechanisms that produce systematic fluctuations in the rate of disease transmission. The aim of this formulation is to maintain the basic structure of a compartmental epidemic model while allowing for transmission to vary periodically.

Let us consider a closed population divided into susceptible, infected, and recovered compartments, denoted by $(\boldsymbol{S(t)}), (\boldsymbol{I(t)}), \boldsymbol{and} (\boldsymbol{R(t)})$, respectively. A periodically forced SIR-type model can be written as

$$\frac{\boldsymbol{dS}}{\boldsymbol{dt}} = \boldsymbol{\Lambda - \beta(t)SI - \mu S},$$
$$\frac{\boldsymbol{dI}}{\boldsymbol{dt}} = \boldsymbol{\beta(t)SI - (\gamma + \mu)I},$$
$$\frac{\boldsymbol{dR}}{\boldsymbol{dt}} = \boldsymbol{\gamma I - \mu R},$$

where $(\boldsymbol{\Lambda})$ is the recruitment rate, \(\mu\) is the natural mortality rate, $(\boldsymbol{\gamma})$ is the recovery rate, and $(\boldsymbol{\beta(t)})$ is the time-dependent transmission coefficient. The transmission function is assumed to be positive and periodic, satisfying

$$\boldsymbol{\beta(t+T) = \beta(t), \quad T > 0}.$$

We call \(T\) the period of transmission variation. The model is therefore different from the classical autonomous SIR system only through the explicit time dependence of the transmission coefficient. This relatively simple modification is enough to give much different qualitative dynamics. A useful representation of periodic transmission is

$$\beta(t) = \beta_0\left[1 + \varepsilon\cos\left(\frac{2\pi t}{T}\right)\right],$$

where (β_0) is the mean transmission level, (ε) is the amplitude of periodic variation, and (T) is the forcing period. If $(0 \le \varepsilon < 1)$, then the transmission coefficient remains positive, and the sinusoidal function is not critical in our theory, since it is convenient to express the recurring transmission patterns. In general, positive periodic functions can also be considered.

The parameter (ε) is important both in a mathematical and biological sense. When $(\varepsilon = 0)$, the system is a constant transmission epidemic. As \(\varepsilon\) increases, the difference between periods of relatively high and low transmission becomes larger. Consequently, the model leads to a continuous transition between an autonomous epidemic system and a strongly periodically forced epidemic system. The parameter \(T\) determines how fast the transmission environment changes. Different combinations of amplitude and period can therefore lead to different epidemic trajectories even if their average transmission levels are comparable.

For a periodically forced epidemic model, the total population $(N(t) = S(t) + I(t) + R(t))$satisfies

$$\frac{dN}{dt} = \Lambda - \mu N.$$

Thus, for a positive initial population,

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}),$$

when $(\mu > 0)$. It follows that the total population remains bounded and approaches the demographic equilibrium (Λ/μ). This property provides an invariant demographic scale for analysing the epidemic subsystem.

The epidemiological variables are required to remain non-negative because negative population sizes have no biological interpretation. Under non-negative initial conditions and a non-negative transmission function, the vector field of the model prevents the solution from crossing the boundary of the biologically feasible region. Consequently, the non-negative region is positively invariant. This property is fundamental for the mathematical validity of the epidemic model.

The model also contains two qualitatively different types of long-term behaviour. When the infection is absent, the system follows the disease-free dynamics. When infection can persist, the system may approach an endemic equilibrium in an autonomous formulation or, under periodic forcing, an endemic periodic state. Thus, periodic transmission changes the natural object of long-term analysis from a fixed equilibrium to a potentially time-dependent periodic solution.

For more complex diseases, the same framework can be extended by introducing additional compartments. For example, an SEIR model incorporates an exposed population and can be represented by adding an incubation compartment between susceptible and infected individuals. Similarly, vaccinated, hospitalized, treated, or temporarily immune populations may be incorporated when required by the biological characteristics of a particular disease. The essential feature remains the same: the transmission process is governed by a positive periodic function of time. An important consequence of this formulation is that the resulting epidemic system is non-autonomous. Standard equilibrium analysis remains useful for identifying reference states, but it is not sufficient to describe all possible long-term behaviours. The periodic coefficient introduces a temporal forcing mechanism that can generate recurrent epidemic patterns, periodic solutions, and changes in stability. Therefore, the subsequent analysis must combine classical epidemic theory with methods for non-autonomous and periodic differential equations. The

mathematical formulation developed in this section provides the basis for analysing the qualitative behaviour of periodically forced epidemic systems. In particular, the positivity and boundedness properties establish the biologically meaningful state space, while the periodic transmission coefficient provides the mechanism through which recurrent epidemic behaviour can arise. The next section examines the resulting epidemic dynamics, with emphasis on disease-free behaviour, threshold conditions, stability, persistence, and extinction.

3. Qualitative Analysis of Epidemic Dynamics

The periodically forced epidemic model introduced in the previous section is a non-autonomous dynamical system since the transmission coefficient is explicitly time-dependent. So the long-term behavior of the infection can't be characterized only by equilibrium analysis for autonomous epidemic models. However, several basic properties—positivity, boundedness, disease-free behavior, and invasion conditions—can be established in periodic context.

Positivity and Boundedness

The biological interpretation of the model requires that the susceptible, infected, and recovered populations remain non-negative for all time whenever their initial values are non-negative. Since the structure of the model guarantees this property, we assume that the transmission coefficient is positive and continuous.

For example, on the boundary $(\boldsymbol{S = 0})$, the susceptible equation is

$$\left.\frac{dS}{dt}\right|_{S=0} = \Lambda > 0,$$

that prevents the susceptible population from becoming negative. Similarly, when \(I=0\),

$$\left.\frac{dI}{dt}\right|_{I=0} = 0.$$

So an infection-free state cannot generate a negative infected population. The recovered population satisfies

$$\left.\frac{dR}{dt}\right|_{R=0} = \gamma I \geq 0.$$

Hence, the non-negative region is positively invariant. This property guarantees that the model remains epidemiologically relevant in the course of evolution.

The total population equation obtained above shows that the population remains bounded when the recruitment and natural mortality rates are finite. Indeed, the biologically feasible region can therefore be regarded as a bounded positively invariant set. These are essential properties to be examined further in the future from a disease persistence point of view.

Disease-Free Dynamics

The disease-free state is the absence of infection, i.e., \(I=0\). In that case, the susceptible population follows the demographic equation

$$\frac{dS}{dt} = \Lambda - \mu S.$$

For $(\boldsymbol{\mu > 0})$, the susceptible population approaches the demographic equilibrium

$$S^0 = \frac{\Lambda}{\mu}.$$

Although the susceptible component approaches a fixed value in the basic demographic formulation, the full periodically forced epidemic system remains non-autonomous because of the transmission coefficient that varies with time. The disease-free state thus provides a reference state to determine if a small introduction of infection can invade the population.

Near the disease-free state, the infected population satisfies the linearized equation

$$\frac{dI}{dt} = [\beta(t)S^0 - (\gamma + \mu)]I.$$

This equation is a direct mathematical representation of the competition between new infections and the combined effects of recovery and natural removal. When the cumulative effect of transmission over successive periods is insufficient to compensate for removal, small infections tend to disappear. If transmission is strong enough, infection can increase from a small initial level.

Epidemic Threshold Under Periodic Transmission

For a constant transmission rate, the basic reproduction number is commonly expressed in terms of the transmission and removal parameters. For periodic forcing, however, the invasion process depends on the whole time profile of $(\beta(t))$ rather than on a single constant value.

For the linearized infected equation, the growth of infection over one complete period (T) is given by the corresponding multiplier

$$Ɗc\{M\} = \exp ⊢[∫ _0\^T ⊢\{\beta(t)S\^0 - (\gamma + \mu)⊣ \}dt⊣].$$

The quantity $(\mathcal{M})$provides a simple threshold interpretation for the scalar infected subsystem. If $(\mathcal{M} < 1)$, a small infection decreases from one period to the next and disease extinction is expected locally. If $(\mathcal{M} > 1)$, the infection can invade the disease-free population. The critical case $(\mathcal{M} = 1)$represents a threshold at which higher-order effects and the detailed nonlinear structure may become important.

This formulation also illustrates an important feature of periodic epidemic systems. If the transmission function is written as

$$\beta(t) = \beta_0\left[1 + \varepsilon\cos\left(\frac{2\pi t}{T}\right)\right],$$

then its average over one complete period is \(\beta_0\). For the scalar linearization considered above, the integral of the periodic component over a full period is zero. Thus, the invasion multiplier depends on the average transmission level in this simplified setting. However, this does not imply that the amplitude and timing of periodic transmission are irrelevant to the complete nonlinear epidemic dynamics. In more general epidemic systems, particularly those involving multiple infected compartments, delays, nonlinear incidence, immunity loss, or periodic interventions, the temporal structure can influence the threshold and stability properties in more complicated ways.

Local Stability of the Disease-Free State

The disease-free state is locally stable when sufficiently small introductions of infection decay with time. In a periodically forced system, this stability is more naturally described through the behavior of the solution over successive forcing periods than through the eigenvalues of a constant Jacobian matrix.

For the scalar infected subsystem, the condition

$$\mathcal{M} < 1$$

provides the local extinction criterion. Equivalently, the average exponential growth rate of the infected population over one forcing period must be negative. When this condition is satisfied, perturbations introduced near the disease-free state diminish over successive periods.

When $(\mathcal{M} > 1)$, the disease-free state loses its local stability with respect to infection. In this situation, a sufficiently small infected population can increase, indicating that the disease has the potential to establish itself within the population. The threshold therefore separates two qualitatively different regimes: infection extinction and epidemic invasion.

Persistence of Infection

Disease persistence refers to the continued presence of infection over long time intervals. In periodically forced models, persistence does not necessarily imply convergence to a constant endemic equilibrium. Instead, the infected population may approach a periodic trajectory whose amplitude and timing reflect the periodicity of transmission.

When the transmission process repeatedly enters phases favorable to infection growth, the disease may be maintained through successive cycles. During periods of high transmission, the infected population may increase, while periods of low transmission may produce substantial declines. If the infection remains sufficiently large to survive the low-transmission phases, recurrent outbreaks can occur indefinitely. Mathematically, persistence is associated with the existence of trajectories that remain separated from the disease-free boundary after sufficiently long time. Establishing such behavior generally requires stronger conditions than local instability of the disease-free state. For more complex models, persistence can be investigated using comparison arguments, invariant-region techniques, monotone dynamical systems, or results from periodic dynamical systems.

Disease Extinction

Extinction occurs when the infected population approaches zero as time tends to infinity. Periodic transmission does not prevent extinction; rather, it modifies the conditions under which extinction occurs. If the average epidemiological growth of infection is less than the removal effect, repeated transmission cycles cannot compensate for recovery and mortality. The infected population then decreases over time. However, short periods of high transmission are not always enough for long-term persistence of infection. The cumulative effect of transmission and removal over subsequent cycles determines if infection can survive indefinitely.

This distinction is especially important when observing seasonal epidemics. A disease could have large outbreaks at high-transmission seasons but be on the verge of extinction if transmission at low-transmission times is not enough to keep the disease spreading.

Influence of Periodic Parameters

The qualitative dynamics of the model depend on several epidemiological quantities. The mean transmission level determines the overall strength of infection, while the amplitude of periodic forcing determines the difference between high and low transmission periods. The forcing period determines the time scale at which these phases occur. The recovery rate has the opposite effect on infection persistence. A larger recovery rate shortens the infectious period and tends to reduce the ability of the disease to maintain transmission. Similarly, demographic removal affects the duration over which individuals remain susceptible or infected.

The interaction between these parameters is more important than any parameter considered in isolation. In particular, the same mean transmission level can produce different epidemic trajectories if the temporal arrangement of transmission is changed. This is especially relevant for multi-compartment systems where exposed and infected populations respond differently to temporal changes in transmission.

Mathematical Significance

We show that periodic transmission transforms the classical epidemic problem into a periodically forced dynamical system. Positivity and boundedness of the model are mathematical checks and the disease-free analysis serves as a reference to understand invasion and extinction. A period-to-period growth multiplier is a natural threshold in the simplest periodically forced epidemic system. For more general models, the threshold can't always be represented by a single elementary integral. So when several infected compartments interact, the linearized infected subsystem becomes a system of periodic differential equations. In such cases, the stability of the disease-free periodic solution can be investigated using the principal Floquet multiplier or other spectral quantities. This is a natural transition from reproduction-number theory to the analysis of non-autonomous epidemic systems. The results of this section lay the mathematical basis for studying recurrent epidemic behaviour. The next section develops this perspective further by investigating periodic epidemic solutions and Floquet theory, and the mechanisms behind periodic transmission for generating sustained recurrent outbreaks.

4. Periodic Epidemic Behaviour and Threshold Dynamics

Periodic transmission can lead to long-term epidemic behaviour that differs fundamentally from convergence to a constant equilibrium observed for many autonomous epidemic models. If the transmission coefficient changes with a fixed period, the epidemic system is constantly subject to the same temporal forcing. As such, the population may develop recurrent epidemic trajectories that are sensitive to the period and amplitude of the transmission pattern.

Periodic Epidemic Solutions

A periodic solution of an epidemic system is a trajectory that satisfies

$$\boldsymbol{x(t+T) = x(t)},$$

where $(\boldsymbol{x(t)})$ is the vector of the epidemiological compartments and \(T\) is the period of the transmission function. Such a solution does not imply that disease incidence is constant. In fact, the infected population may increase and decrease continuously and be back in the same state after each transmission cycle. Periodic epidemic solutions provide a natural mathematical representation of recurrent outbreaks. In high-transmission phases, the number of new infections is higher, while in low-transmission periods recovery and removal are more dominant. If the balance between these processes is still similar from one cycle to another, then the system can follow a stable periodic path. This behaviour is qualitatively different from an endemic equilibrium. In an endemic equilibrium, the compartment sizes are constant over time. But under periodic forcing, the long-term endemic state can fluctuate in a periodic fashion. Therefore, it is important to question whether the endemic state depends on a relatively independent epidemic environment or on periodic variations.

Role of the Forcing Period

The period (T) determines the time scale of the external forcing. A short period of time is associated with rapid changes in transmission, whereas a long period of time is associated with longer high and low transmission. The epidemic response is determined not only by the magnitude of these changes but also by the time scale of the change in transmission compared to other epidemiological processes such as infection and recovery. If the transmission environment is much more rapidly changing than the typical epidemic time scale, the population may respond mostly to an averaged transmission effect. On the other hand, if the forcing period is comparable to the characteristic infectious time scale, the epidemic may respond strongly to individual phases of the transmission cycle.

This interaction between epidemiological and forcing time scales can result in large variations in epidemic incidence. Therefore, the period of transmission should be considered as an important model parameter rather than only a mathematical specification of periodicity.

Periodic Forcing and Recurrent Outbreaks

One of the most important consequences of periodic transmission is the possibility of recurrent outbreaks. Suppose that the transmission coefficient moves from high to low. In a high-transmission phase, the effective rate of infection is larger than the effective rate of removal, so that the number of infected patients will increase. And when the transmission rate is decreasing, recovery and removal will be dominant, leading to a decline in infectivity.

If enough susceptible individuals are available for the next transmission cycle, the process can repeat. The incidence curve may therefore have a sequence of epidemic peaks and periods of decreased infection. Such recurrent behaviour does not necessarily mean the pathogen is not introducing itself independently of the time of transmission but rather the periodic nature of the transmission environment. The mathematical difference between transient recurrence and asymptotic periodic behaviour is important. A finite number of epidemic peaks may be transient dynamics before the system approaches a disease-free or periodic state. Persistent recurrence, on the other hand, requires the long-term maintenance of infection and is associated with the existence and stability of a proper periodic path.

Floquet-Theoretic Interpretation

For a periodically forced nonlinear epidemic system, stability near a periodic solution can be considered by examining perturbations around that solution. The resulting variational system has periodic coefficients and can be represented in the form

$$\frac{d\xi}{dt} = A(t)\xi, \qquad A(t+T) = A(t),$$

where $(A(t))$ is the Jacobian matrix evaluated along the periodic solution.

Floquet theory provides a mathematical framework for studying such systems. The evolution of perturbations over one complete period is represented by a monodromy matrix. Its eigenvalues are called Floquet multipliers. These quantities determine whether perturbations increase or decrease from one period to the next. In order to be stable for a periodic solution, the Floquet multipliers need to lie inside the unit circle. A multiplier with magnitude greater than one indicates growth of a perturbation and instability of the periodic trajectory. The boundary case, in which a multiplier has magnitude equal to one, represents a critical situation in which higher-order analysis may be required.

The Floquet perspective is particularly useful because it replaces the eigenvalue analysis of a constant Jacobian with a period-to-period stability analysis. It therefore provides a natural mathematical extension of classical local stability theory to periodically forced epidemic systems.

Threshold Dynamics in Periodic System

The epidemic threshold can be understood as the boundary between infection extinction and invasion success. In a periodically forced system, this threshold is determined by the growth of infection over repeated transmission cycles and not by its behaviour at a single instant of time. For a simple scalar infected equation, the period multiplier introduced in the first section provides this interpretation. For a multi-compartment epidemic model, however, the infection may pass through many epidemiological stages. An infected individual may first enter an infectious compartment and then recover, with each stage affected by different transition rates. For such a system, the threshold is naturally related to the dominant Floquet multiplier of the linearized infected subsystem. If the dominant multiplier is below one, the perturbations in the infected compartments will diminish from one cycle to the next, corresponding to local disease extinction. If it exceeds one, infection can increase over successive cycles and invade the disease-free periodic environment. This criterion provides a generalized threshold principle for non-autonomous epidemic systems.

Periodic Transmission Versus Average Transmission

An important theoretical question is whether the average transmission coefficient is sufficient to predict epidemic behaviour. For the simplest scalar linearized model we have considered before, integration over one complete period can cause the zero-mean periodic component to cancel out. However, this does not generally hold for more complex epidemic systems. In multi-compartment models, nonlinear incidence functions, immunity loss, time delays, periodic vaccination, and other mechanisms can create interactions between the transmission pattern and the internal epidemic dynamics. The timing of high-transmission intervals can then influence the epidemic trajectory even if the average transmission level remains unchanged. This is a key point of modelling. Replacing a periodic transmission coefficient by its average value can simplify the analysis but it is possible that the autonomous model in turn cannot reproduce some important temporal features of the original system. In particular, epidemic peak timing, peak magnitude, frequency, recurrent incidence, and transient persistence of waves can depend on the transmission structure.

Influence of Periodic Amplitude

The amplitude of transmission variation determines the difference between high and low transmission phases. For a periodic transmission function of the form

$$\boldsymbol{\beta(t) = \beta_0 \left[1 + \varepsilon \cos\left(\frac{2\pi t}{T}\right)\right]},$$

the parameter $(\boldsymbol{\varepsilon})$ determines this variation.

When $(\boldsymbol{\varepsilon})$ is small, the periodically forced system is close to the corresponding constant-transmission model, but when $(\boldsymbol{\varepsilon})$ increases, the transmission fluctuations start to be more severe and the epidemic peaks and troughs are much more pronounced than before. The effect of increasing the amplitude is not only about an increase in the average transmission, it is also about a change in the time structure of the epidemic process.

In a nonlinear epidemic system, a sufficiently strong periodic forcing can also affect the stability of periodic solutions and lead to a transition between qualitatively different dynamical regimes. The exact behaviour depends on the model structure and parameter values, and thus requires mathematical analysis rather than on the average transmission coefficient.

Implications for Epidemic Control

The periodic structure of transmission has direct implications for epidemic intervention. If periods of high transmission can be identified, vaccination, treatment, testing, isolation, or behavioural interventions may theoretically be adjusted according to the transmission cycle.

A periodic intervention can be represented mathematically by allowing one or more epidemiological parameters to vary with time. For example, vaccination coverage or treatment intensity may be increased during anticipated high-transmission periods. Such ways also introduce an additional periodic forcing into the epidemic system and can change the stability of the disease-free or endemic periodic state.

The mathematical objective is not to reduce transmission at every instant, but to see how the timing, duration, and intensity of interventions collide with the natural periodicity of disease transmission. This provides a framework for studying time-dependent control strategies in a dynamical-systems context.

Overall Mathematical Perspective

The periodic epidemic dynamics show that temporal variability can also be a natural feature of the mathematical structure of an infectious disease model. Periodic solutions provide a natural description of recurrent endemic behaviour and Floquet theory is a good way to study their stability. Period-to-period growth provides a threshold to disease invasion and extinction. The results also show that epidemic behaviour is dependent on biological processes and time-dependent forcing. Transmission amplitude, forcing period, recovery, demographic turnover, and compartmental structure all play a part in the dynamics. As a result, periodically forced epidemic models can be used as a bridge between the classical autonomous epidemic models and more realistic time-dependent models of infectious disease transmission.

5. Biological Interpretation and Modelling Implications

The mathematical analysis in the previous sections shows that periodic variation in transmission can have a highly influential effect on the time behaviour of an epidemic. The introduction of a time-dependent transmission coefficient provides a mathematical way to describe recurrent changes in the spread of diseases and allows epidemic models to go beyond the assumption of a constant transmission environment. The mathematical results are discussed in this section from a biological and epidemiological perspective and have implications for the construction and analysis of epidemic models.

Seasonal Transmission

Seasonality is one of the most important mechanisms which can generate periodic transmission patterns. Environmental conditions such as temperature, humidity, rainfall, and daylight can have an effect on pathogen survival, vector abundance, host susceptibility and human contact patterns. These factors may lead to periods where disease transmission becomes relatively more or less favourable. A periodically varying transmission coefficient is a simple mathematical representation of such effects. The amplitude of the periodic function describes the strength of seasonal variation and its period is the characteristic time

scale of the environmental cycle. In most cases, an annual period is appropriate but other periods may be relevant depending upon the biological system. Seasonality can thus transform an epidemic from a transient outbreak into a series of recurrent outbreaks. The mathematical model does not require every outbreak to be exactly the same size, though. The populations of susceptible people, demographic turnover and previous history of infection can vary in the subsequent peak in the epidemic even if the external forcing is periodic.

Human Behaviour and Contact Patterns

The temporal variation in transmission is not necessarily due to environmental factors. Human behaviour may also create periodic changes in contact rates. School calendars, working schedules, holidays, mass gatherings, travel patterns and seasonal social activities may systematically alter the frequency and structure of interpersonal contacts. From a mathematical point of view, these mechanisms can be incorporated into the transmission coefficient without modelling every behavioural process explicitly. This is a practical compromise between biological realism and mathematical tractability. But this should be interpreted carefully because the same periodic transmission pattern can arise from several different underlying mechanisms. The model describes the resulting transmission environment rather than finding a single biological cause of the periodicity. If data is available, parameter estimation and model comparison can help determine whether environmental, behavioural, demographic, or combined mechanisms are more appropriate.

Recurrent Epidemic Outbreaks

The periodic framework provides a mathematical explanation for recurrent epidemic outbreaks. During periods with increased transmission, the infected population can grow more rapidly. The low transmission phase can reduce the burden of infection through recovery and removal. If enough susceptible individuals are still available, a new outbreak may become prevalent in the middle of a high transmission phase. This mechanism shows why repeated epidemic peaks do not necessarily require repeated external introductions of infection. Recurrence may emerge from the interaction between periodic forcing and the internal dynamics of the host population. The magnitude and timing of subsequent outbreaks depend on the transmission periodicity and epidemiological time scale of infection, recovery, immunity and demographic replacement. Therefore, periodic epidemic models can give insights into the regularity and variability of recurrent outbreaks.

Importance of the Susceptible Population

The susceptible population plays a critical role in determining the response of an epidemic to periodic forcing. During a high-transmission phase, new infections can dramatically reduce the susceptible population. During low-transmission periods, the infected population may decline and the susceptible population is gradually restored through demographic recruitment or loss of immunity, depending on the model structure. This interaction leads to an internal feedback mechanism. Periodic transmission alone does not determine the epidemic trajectory; its effect depends on the state of the population at the time each transmission phase occurs. Thus, two cycles with similar transmission intensity can lead to different numbers of infections if they begin with different susceptible populations. This feedback is one reason why periodically forced nonlinear epidemic models can display richer dynamics than models based only on a constant average transmission rate.

Implications for Epidemic Thresholds

The threshold concept is still integral to epidemic modelling but periodic transmission changes its meaning. In an autonomous model, a reproduction number is generally associated with a fixed epidemiological environment. In the presence of periodic forcing, the relevant invasion process takes place across a changing environment. The mathematical analysis implies that epidemic thresholds should be interpreted in terms of the whole transmission cycle. For simple systems, an average transmission quantity may be a good approximation. For more complex systems, the threshold may depend on the interaction between multiple compartments and on the timing of transmission and removal. This is especially important when comparing mathematical models. A constant transmission approximation may reproduce the average incidence of a periodically forced system, but does not reproduce its temporal structure. Thus, agreement in average disease burden does not imply agreement in epidemic dynamics.

Periodic Intervention Strategies

The same mathematical framework can be used to investigate interventions that vary over time. Vaccination campaigns, treatment programmes, public health restrictions, screening activities, and other interventions may be intensified during periods when transmission is expected to increase. In addition, periodic intervention introduces an additional time-dependent component of the epidemic system. Its benefit is not only dependent on the intensity of the intervention but also its timing relative to the natural transmission cycle. An intervention that is applied immediately before or during a high transmission phase may have a different mathematical effect than the same intervention applied during a low transmission phase. This observation motivates the study of synchronized intervention strategies in which control measures are directed to the periodic structure of disease transmission. Such models can be used to investigate whether temporal allocation of limited intervention resources can change the stability of the disease-free state or reduce recurrent epidemic peaks.

Model Assumptions and Limitations

While periodic transmission can be a useful extension of classic epidemic models, there are some assumptions that should be recognized. By assuming a periodic function for transmission, it is assumed that the transmission is regular enough to be approximated by a known function. In real epidemic systems, there may be large fluctuations in environmental conditions, behavioural changes, demographic changes, and unexpected interventions. Moreover, the transmission coefficient may depend on several factors and not on a single periodic mechanism. A simple sinusoidal function captures regular variation but may not capture asymmetric seasonal patterns or multiple transmission peaks in one period. Another limitation is that many compartmental models assume homogeneous mixing. In real populations, individuals in different age, occupation, geography, immunity, behaviour and contact structure differ. In addition, this can greatly affect the impact of periodic transmission. Therefore, the periodically forced compartmental model should be considered as a theoretical baseline and not a complete description of every epidemiological system.

Broader Mathematical Implications

The framework developed in this paper demonstrates the importance of considering time dependence when constructing mathematical models of infectious diseases. Periodic forcing provides a relatively

simple mechanism through which non-autonomous dynamics can be introduced, yet still allow rigorous mathematical analysis.

The combination of epidemic modelling and periodic dynamical systems theory opens up a wide range of opportunities to study recurrent outbreaks, seasonal persistence, threshold transitions, periodic endemic states and time-dependent intervention. The framework can also be extended to models with multiple pathogens, age structure, spatial movement, temporary immunity, vaccination and behavioural feedback. More sophisticated formulations can incorporate periodic transmission with stochastic effects, spatial diffusion, fractional operators, or data-driven parameter estimation. Such extensions can provide more realistic models of infectious disease systems while maintaining the fundamental mathematical argument that transmission and epidemiological processes may change systematically over time.

Overall Interpretation

The mathematical results show that periodic transmission should not be viewed as a small modification of a constant transmission parameter. It changes the structure of the epidemic system from autonomous to non-autonomous and can change the nature of its long-term behaviour. Disease dynamics might be characterized by recurrent oscillations and periodic states rather than constant endemic levels. The biological interpretation is thus that the epidemic behavior depends not only on how strongly a pathogen is transmitted but also on when it is transmitted. The time of transmission, susceptible availability, recovery, demographic processes, and intervention all influence the epidemic trajectory. Periodically forced epidemic models can therefore provide a theoretical framework for connecting temporal distributions in biological transmission to the qualitative dynamics of infectious diseases.

6. Extensions and Future Research Directions

The periodically forced epidemic model proposed in this paper provides a simple mathematical framework for investigating recurrent changes in disease transmission. While the model captures a key feature of epidemic systems, many extensions may be considered to incorporate more biological mechanisms and to make it more appropriate for the more complex infectious disease processes.

Periodic Transmission in Multi-Compartment Models

The SIR formulation is a good starting point, but many infectious diseases require additional epidemiological compartments. Exposed, hospitalized, vaccinated, treated, or temporarily immune individuals can be taken into account to represent disease-specific characteristics. For example, in a SEIR-type model, the exposed compartment introduces an incubation period between infection and infectiousness.

Periodic transmission in such multi-compartment systems can lead to more complex phase relationships between transmission forcing and epidemic incidence. The resulting linearized infected subsystem is generally composed of several coupled periodic differential equations, so the Floquet-theoretic analysis is very relevant.

Periodic Vaccination and Treatment

An important extension is to allow the intervention parameters to vary periodically. For instance, vaccination programs may be carried out during certain seasons, and treatment and screening intensity

may increase during anticipated epidemic periods. Such approaches can be depicted by time-dependent vaccination or treatment rates.

The mathematical problem is then to know how the timing and intensity of intervention affect the stability of the disease-free periodic state. This framework can also be used to assess whether timed interventions can better suppress recurrent outbreaks than interventions of the same intensity but with different temporal characteristics.

Multiple Strains and Variant Dynamics

Many infectious diseases have multiple strains or variants that may differ in transmissibility, recovery properties, or immune escape. Extending the periodic framework to multi-strain systems would allow the interaction between temporal forcing and competition among pathogen variants to be investigated.

The effects of periodic transmission on different strains will be different when infectious periods and transmission characteristics are distinct. This model would likely offer a theoretical basis for studying strain replacement, coexistence, and recurrent dominance in seasonally varying transmission environments.

Age-Structured and Heterogeneous Populations

The assumption of homogeneous mixing simplifies mathematical analysis but does not adequately represent populations with strong differences in contact patterns. Age-structured models can distinguish transmission among children, adults, and older people. Network and activity-based models can capture heterogeneous contact structures.

With periodic transmission and age-dependent contact patterns, different epidemic responses can be observed among groups. Mathematical analysis of such systems could thus provide more detailed insight into the mechanisms underlying seasonal or recurrent outbreaks.

Spatial and Metapopulation Extensions

Temporal periodicity can also be combined with spatial movement. Populations distributed across different geographical locations might have different transmission patterns, while travel and migration connect these regions.

A metapopulation model with periodic transmission could be used to investigate how local seasonal forcing interacts with movement between regions. More general spatial epidemic models could include diffusion or directed movement in addition to periodic transmission, together providing a temporal and spatial framework.

Stochastic Periodic Epidemic Models

Deterministic models assume that epidemic trajectories are completely determined by the initial conditions and model parameters. However, at low population sizes, random events can strongly influence whether an infection becomes established or disappears.

We can introduce stochasticity into a periodically forced epidemic model to allow random demographic and transmission events to be included. Such models might test the likelihood of invasion, extinction, and recurrent epidemics in a periodically varying transmission environment.

Delayed and Fractional Epidemic Models

Biological processes often involve delays. Incubation periods, immune responses, treatment delays, and behavioural adaptation cannot always be described well by instantaneous interactions. Delay differential equations are thus a natural extension of the periodic framework.

Fractional epidemic models provide a different approach by incorporating memory effects into disease dynamics. Combining fractional operators with periodic transmission might be particularly useful for studying systems in which the present epidemic state depends on the history of infection or recovery processes.

Data-Driven Periodic Transmission

An important direction for future research is to estimate periodic transmission functions from epidemiological observations. Rather than prescribing a sinusoidal transmission coefficient, empirical incidence and mobility data could be used to estimate the temporal structure of transmission.

Such an approach would enable the mathematical model to distinguish between regular seasonal forcing and irregular temporal fluctuations. Parameter estimation, inverse problems, and data assimilation could thus connect theoretical periodic epidemic models with observed disease dynamics.

Mathematical Analysis of Long-Term Behaviour

Future theoretical work can study stronger conditions for the existence, uniqueness, and stability of periodic epidemic solutions. In particular, the relationship between the dominant Floquet multiplier and the generalized reproduction numbers of a multi-compartment model needs to be studied more mathematically.

Other important questions include global stability, uniform persistence, bifurcation of periodic solutions, and transitions between qualitatively different epidemic regimes. These problems are possible to relate classical epidemic theory to the theory of non-autonomous dynamical systems.

Future Research Perspective

The periodic transmission framework provides a foundation that can be progressively extended without changing its basic mathematical principle. The idea is that epidemiological parameters might have an intrinsic temporal structure and such structure can affect both outbreaks and long-term disease behaviour.

So future research can move towards integrated models containing periodic transmission with demographic heterogeneity, spatial movement, stochasticity, delays, changes in behaviour, and empirical data. Such models would provide a more detailed mathematical description for infectious diseases and still retain the analytical advantage of dynamical systems theory.

The development of these extensions is especially important because real epidemic systems rarely operate under completely constant environmental and behavioural conditions. A mathematical framework that accounts for temporal variability can thus offer a more flexible basis for understanding disease persistence, recurrent outbreaks, and time-dependent epidemic control.

CONCLUSION

Periodic variation in disease transmission is an important mathematical mechanism for understanding the recurrent and time-dependent behaviour of infectious diseases. In contrast to traditional epidemic models with constant transmission parameters, periodically forced models account for systematic temporal changes due to seasonal conditions, human behaviour, environmental factors and recurring patterns of population contact.

In this paper we introduced a mathematical framework for understanding the processes of epidemic dynamics under periodic transmission patterns. We developed a periodically forced SIR-type model where the transmission coefficient is a positive periodic function. We showed that this system retains the basic epidemiological properties of positivity and boundedness, thus it can be regarded as a biologically relevant setting for qualitative analysis.

We specifically considered the disease-free state and the conditions for infection invasion and extinction. We showed that in the periodically changing environment epidemic thresholds should be understood in terms of full transmission cycles instead of individual points in time. For the basic scalar infected subsystem a period-to-period growth multiplier provides a direct answer for local disease extinction or invasion. More general multi-compartment systems are naturally periodic linear systems and Floquet theory is the best model for stability analysis.

An important consequence of periodic transmission is that there may be a pattern of recurrent epidemic behaviour. Rather than a constant endemic equilibrium an epidemic system may follow, that is, be on a periodic path in which infection increases and decreases with the transmission cycle. The amplitude and period of the forcing is very important, as well as the recovery rate, demographic turnover and susceptibility. The temporal organization of transmission is the reason for the time-series behaviour of the epidemic.

The analysis also highlights an important limitation of replacing periodic transmission by its average value. While averaging may be useful for some simplified systems, nonlinear interactions between epidemiological compartments can affect the time structure of transmission and thus the peaks, persistence, stability, and recurrent outbreaks of epidemics. Hence, models with periodic forcing can provide information not captured in the corresponding autonomous models.

From the biological perspective, the framework is applicable for seasonal and recurrent infectious diseases and also to a number of time-dependent interventions, such as vaccination, treatment and screening. The mathematical formulation thus provides the basis to study not just the natural variation of epidemic infections but also the effects of interventions that are synchronized with the transmission cycle.

The following extensions remain open to future study. These include age-structured and multi-strain models, spatial and metapopulation systems, stochastic epidemic models, delayed and fractional formulations, periodic control strategies and data-driven estimation of transmission functions. Further mathematical work on global stability, uniform persistence, bifurcation of periodic solutions and generalized epidemic thresholds may offer more insight into the long term behaviour of periodically forced epidemic systems.

In general, our study shows that using periodic transmission in epidemic models provides a mathematically meaningful extension of classical epidemiological theory. It connects the temporal variation of transmission of disease with the qualitative dynamics of infection and provides a theoretical framework for understanding disease extinction, persistence, recurrent outbreaks and periodic epidemic behaviour.

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