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Reaction-Diffusion Equations: Mathematical Theory, Stability and Pattern Formation

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Abstract:
At the academic level, a basic approach to mathematic modeling of reactions and diffusion processes is given by the reaction-diffusion equations. This work presents an analysis of reaction-diffusion equations, potentially leading to some interesting results in studying the phenomenon of diffusion instability,which is manifested in formation of spatial patterns. The new system of equations is provided for analysis; the stability analysis of equilibria of selected systems is performed employing linearization and spectral analysis. In this paper, the analysis of the influence of diffusion on stability spectra is conducted with aim to identify the conditions, under which equilibrium point being stable from the viewpoint of stability becomes unstable due to diffusion process. The Turing instability mechanism is studied as a key mechanism for appearance of non-homogeneous stationary structures. In the current research project, typical examples of reaction-diffusion models have been used to illustrate the theoretical framework. In this case, the Schnakenberg model has served as one of the examples included in the research focusing on the identification of equilibrium states and the conditions for instability and calculating characteristic spatial disturbances. The results of the research demonstrate the process of the smallest spatial disturbances' evolution and formation of stable spatial structures. The analysis reveals the relationships connecting chemical reactions, diffusion coefficients, stability of systems, and pattern formation process.

Keywords: Reaction-diffusion equations; Stability analysis; Diffusion-driven instability; Turing instability; Pattern formation; Equilibrium solutions; Linearization; Dispersion relation; Spatial patterns; Schnakenberg model.

1. Introduction

Reaction-diffusion equations are an important type of partial differential equations which are useful in modeling the changes of interacting quantities present in a certain spatial area. Their mathematical form is due to the two important processes that act at the same time: reaction, which is defined as the process of local interaction, transformation, and production, and diffusion, which refers to the spreading of the interacting quantities over the space. The combination of the two processes leads to a variety of dynamic processes that cannot be explained with the help of each of the processes separately. In particular, the reaction-diffusion systems show stable homogeneous states, spatially heterogeneous states, moving waves, oscillating behaviors, and stationary spatial patterns. Therefore, the study of the phenomena has become one of the significant research issues in the field of nonlinear partial differential equations.

A general reaction-diffusion system for an mm-component vector-valued function $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) = (\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_m})^{\boldsymbol{T}}$ defined on a spatial domain $\boldsymbol{\Omega} \subset \boldsymbol{R^n}$ can be expressed as

$$\frac{\partial u}{\partial t} = D\Delta u + F(u), \qquad x \in \Omega, \quad t > 0,$$

where DD denotes the diffusion matrix, Δ is the Laplace operator, and $F: R^m \to R^m$ represents the nonlinear reaction kinetics. In the simplest diagonal case,

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix},$$

with $d_i > 0$ denoting the diffusion coefficient of the i-th component. The mathematical behaviour of the system depends not only on the nonlinear structure of F, but also on the relative values of the diffusion coefficients, the geometry of the spatial domain and the initial and boundary conditions. An important mathematical question is whether the equilibrium state is stable when spatial diffusion is incorporated into the system. Consider a spatially homogeneous equilibrium u* satisfying

$$F(u^*) = 0.$$

In the corresponding spatially homogeneous ordinary differential equation

$$\frac{du}{dt} = F(u),$$

the equilibrium u(Ω) may be locally asymptotically stable. However, diffusion can change the stability of this equilibrium because the spatial perturbations evolve with different spatial modes. An equilibrium that is stable with respect to homogeneous perturbations can become unstable with respect to different spatial modes. This is known as diffusion-driven instability and is the mathematical basis of Turing's theory of pattern formation. The difference between temporal stability and spatial stability is fundamental in the study of reaction-diffusion systems. Without diffusion, the local dynamics are entirely determined by the reaction terms and can be studied using the Jacobian matrix of the reaction kinetics at equilibrium. If

$$J = DF(u^*)$$

denotes this Jacobian, the eigenvalues of J determine the linear stability of the homogeneous equilibrium in the corresponding reaction system. However, if diffusion is introduced, then also perturbations have to be resolved according to their spatial structure. This gives rise naturally to a spectral problem in which the reaction Jacobian and diffusion matrix interact with the eigenvalues of the spatial Laplacian. The resulting dispersion relation determines the growth or decay rate of each spatial mode and provides a systematic way to identify unstable wavelengths.

For a two-component system,

$$\frac{\partial u}{\partial t} = d_u \Delta u + f(u, v),$$
$$\frac{\partial v}{\partial t} = d_v \Delta v + g(u, v),$$

the stability problem can be investigated by linearizing the reaction terms around an equilibrium $(u*, v*)(u^*, v^*)$. Writing

$$u = u^* + \tilde{u}, \qquad v = v^* + \tilde{v},$$

and retaining only the first-order terms gives a linear reaction-diffusion system governed by the Jacobian

$$J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(u^*, v^*)}.$$

The interaction between JJ and the diffusion matrix

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{d_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{d_v} \end{pmatrix}$$

determines the evolution of spatial perturbations. By expanding the perturbation in eigenfunctions of the Laplacian, the infinite-dimensional stability problem can be decomposed into a family of finite-dimensional eigenvalue problems with the spatial wave numbers. This is a convenient mathematical framework to obtain the necessary and sufficient conditions for diffusion-driven instability. The appearance of spatial patterns is very important because it indicates spatial structure is possible to appear from a very homogeneous state at the start without an externally imposed spatial model. In the mathematical setting, a small perturbation of an equilibrium can be decomposed into spatial modes. If all modes decay, the equilibrium is still stable and the system returns to its homogeneous state. If some of the spatial modes have positive growth rates, the growth rate increases while others decay and the resulting pattern is not uniform. The most unstable mode plays an important role in defining the spatial scale of the emerging pattern. This is the classic Turing mechanism. In a suitable two-component reaction-diffusion system, the reaction kinetics stabilize the homogeneous equilibrium in the absence of diffusion, but the differential diffusion rate can destabilize some of the spatial modes. So diffusion is not necessarily just a smoothing mechanism. Although diffusion tends to reduce spatial gradients, when it combines with nonlinear reaction kinetics, it can destabilize homogeneous states and generate structured spatial distributions. This paradoxical behavior is one of the basic mathematical properties of reaction-diffusion systems. But the study of pattern formation requires that there are more than one connection in the mathematical process. First, the equilibrium states of reaction kinetics need to be identified. Second, their stability in the absence of diffusion must be determined. Third, spatial perturbations must be introduced and the dispersion relation derived. Finally, the range of spatial wave numbers for which the real part of the associated growth rate is positive must be determined. These steps provide a systematic way from the local reaction dynamics to the emergence of spatial patterns. Another important aspect is the role of boundary conditions. A reaction-diffusion equation cannot be considered independent of the spatial domain and boundary conditions. Common choices are homogeneous Neumann conditions,

$$\boldsymbol{\nabla ui \cdot n = 0 on\ \partial\Omega, \nabla u_i \cdot n = 0} \quad \text{on } \boldsymbol{\partial\Omega},$$

which represent zero flux across the boundary, and homogeneous Dirichlet conditions,

$$\boldsymbol{ui = 0 on\ \partial\Omega. u_i = 0} \quad \text{on } \boldsymbol{\partial\Omega}.$$

The eigenvalues and eigenfunctions of the Laplacian are sensitive to these conditions and will affect the spatial modes of the system. Therefore, the shape of the domain and boundary conditions can affect the onset and structure of pattern formation. From a theoretical point of view, reaction-diffusion systems are the natural connection between nonlinear dynamical systems and partial differential equations. The reaction terms determine the local phase space dynamics while diffusion terms are spatial coupling and transform the problem into an infinite-dimensional dynamical system. This allows us to apply the techniques from matrix theory, spectral analysis, bifurcation theory, stability theory, and numerical analysis in one unified mathematical framework. Among the models used to study these phenomena, the Schnakenberg reaction-diffusion system is a particularly useful example as it has a simple nonlinear structure and is characterized by diffusion-driven pattern formation for proper parameter regimes. With such a model, we can explicitly find equilibrium states, find the Jacobian matrix, define stability conditions, obtain the dispersion relation, and study the onset of Turing instability. They are thus a good bridge between abstract theoretical discussion and computational visualization. In spite of the extensive development of reaction-diffusion theory, there are still several analytical problems that remain. The

stability analysis is much harder with more than two interacting components, non-diagonal diffusion matrices, cross-diffusion, nonlocal interactions, heterogeneous domains, or nonlinear boundary conditions. In the same way, nonlinear stability and the behaviour of solutions after the onset of instability cannot always be deduced simply from linear analysis. Numerical methods are thus useful to study the nonlinear evolution of unstable modes and spatial structure formation after the instability. The purpose of this paper is to develop a comprehensible mathematical framework for reaction-diffusion systems, with particular attention to equilibrium states, linear stability, diffusion-driven instability, and spatial pattern formation. The general mathematical formulation of reaction-diffusion systems is first established with the corresponding initial and boundary conditions and linearization around homogeneous equilibria. Then the stability of the corresponding reaction system is compared with the stability of the full reaction-diffusion system. The dispersion relation is derived in particular, along with Turing instability. The theoretical analysis is then described with typical reaction-diffusion models, and the diffusion coefficients, unstable spatial modes, and the patterns are shown. Numerical illustrations are also displayed in this section to further illustrate the analysis and show the development of the spatial structures from small perturbations. The next section is devoted to the mathematical formulation and basic characteristics of reaction-diffusion systems. The subsequent section elaborates on the stability analysis and creates the conditions for diffusion-driven instability. The pattern-formation section analyzes the representative model and numerical behavior for unstable spatial modes. Finally, the main mathematical observations are summarized and possible directions for further study are discussed.

2. Mathematical Formulation and Basic Theory

Reaction-diffusion systems combine local nonlinear interactions with spatial transport and provide a natural mathematical framework for studying the evolution of multiple interacting quantities in space and time. The mathematical formulation of such systems is based on the balance between the rate of local change, the flux produced by diffusion, and the reaction mechanisms acting within the medium. In this section, a general reaction-diffusion system is formulated, followed by the characterization of equilibrium states, linearization around homogeneous equilibria, and the role of initial and boundary conditions.

2.1 General Reaction-Diffusion System

Let $\boldsymbol{\Omega \subset R^n}$be a bounded spatial domain with sufficiently smooth boundary $\boldsymbol{\partial\Omega}$, and let $\boldsymbol{t \geq 0}$ denote time. Consider an mm-component state vector

$$\boldsymbol{u(x,t)} = \begin{pmatrix} \boldsymbol{u_1(x,t)} \\ \boldsymbol{u_2(x,t)} \\ \vdots \boldsymbol{u_m(x,t)} \end{pmatrix}.$$

A general reaction-diffusion system can be written as

$$\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial t}} = \boldsymbol{D\Delta u + F(u)}, \qquad \boldsymbol{x \in \Omega}, \quad \boldsymbol{t > 0},$$

where $\boldsymbol{D \in R^{m\times m}}$ is the diffusion matrix, \Delta denotes the Laplacian operator, and

$$\boldsymbol{F(u)} = \begin{pmatrix} \boldsymbol{f_1(u)} \\ \boldsymbol{f_2(u)} \\ \vdots \boldsymbol{f_m(u)} \end{pmatrix}$$

represents the nonlinear reaction kinetics.

When diffusion acts independently on each component, the diffusion matrix is diagonal,

$$\boldsymbol{D = \backslash operatorname{diag}(d_1, d_2, \ldots, d_m)},$$

where $d_i > 0$ is the diffusion coefficient associated with u_i. The system then takes the componentwise form

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(u_1, \dots, u_m), \qquad i = 1, \dots, m.$$

The reaction term f_i determines the local temporal evolution of the i-th component, whereas $d_i \Delta u_i$ describes its spatial redistribution. The competition between these two mechanisms is responsible for many of the qualitative behaviours observed in reaction-diffusion systems.

For a scalar equation, the model reduces to

$$\frac{\partial u}{\partial t} = d \Delta u + f(u).$$

Although the scalar model is mathematically important, diffusion-driven instability in the classical Turing sense requires interaction between at least two components with appropriately different diffusion rates. Therefore, the principal stability analysis in this paper focuses on coupled systems.

For a two-component system, the general form is

$$\frac{\partial u}{\partial t} = d_u \Delta u + f(u, v),$$
$$\frac{\partial v}{\partial t} = d_v \Delta v + g(u, v).$$

Here u and v represent two interacting state variables, while d_u and $d_v$denote their diffusion coefficients. The functions ff and gg describe the local nonlinear interactions between the components.

An important mathematical feature of this formulation is that diffusion is represented by the Laplacian. In Cartesian coordinates,

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Thus, the diffusion term depends on spatial curvature. Regions of high concentration tend to redistribute toward regions of lower concentration, thereby smoothing spatial gradients. However, when diffusion acts simultaneously on interacting components with different rates, its interaction with the reaction kinetics can produce instability rather than simple homogenization.

2.2 Initial and Boundary Conditions

A reaction-diffusion system must be supplemented by appropriate initial and boundary conditions. The initial state is specified by

$$u(x, 0) = u_0(x), \qquad x \in \Omega,$$

where u_0 represents the initial spatial distribution.

Two boundary conditions are particularly important in the analysis of reaction-diffusion systems. The first is the homogeneous Neumann, or zero-flux, condition

$$\nabla u_i \cdot n = 0, \qquad x \in \partial\Omega,$$

where n\mathbf n denotes the outward unit normal vector to the boundary. In vector notation,

$$\nabla ⊣ \mathbf{b} u \, ⊣ \mathbf{b} n = 0 \qquad "\{on\}\partial\Omega.$$

This condition prevents net flux across the boundary and is frequently used when the domain is regarded as spatially closed.

The second common choice is the homogeneous Dirichlet condition

$$u_i = 0, \qquad x \in \partial\Omega.$$

The choice of boundary condition is mathematically significant because it determines the eigenvalues and eigenfunctions of the Laplacian. Since the stability of spatial modes depends on these eigenvalues, the boundary conditions can influence which spatial patterns are permitted.
For the analysis of Turing instability, homogeneous Neumann conditions are particularly convenient. Under these conditions, the eigenfunctions of the Laplacian form a suitable basis for representing spatial perturbations. Consequently, the stability problem can be decomposed into independent spatial modes.

2.3 Homogeneous Equilibrium States

A spatially homogeneous equilibrium is a constant vector

$$\boldsymbol{u}^* = (\boldsymbol{u}_1^*, \boldsymbol{u}_2^*, \dots, \boldsymbol{u}_m^*)^T$$

satisfying

$$\boldsymbol{F}(\boldsymbol{u}*) = \mathbf{0}$$

Since $\boldsymbol{u}^*$is independent of xx,

$$\boldsymbol{\Delta u}* = \mathbf{0}$$

Therefore,

$$\boldsymbol{D\Delta u}* + \boldsymbol{F}(\boldsymbol{u}*) = \mathbf{0},$$

and hence $\boldsymbol{u}^*$ is also an equilibrium of the complete reaction-diffusion system.
For the two-component system,

$$\boldsymbol{f}(\boldsymbol{u}^*, \boldsymbol{v}^*) = \mathbf{0}, \qquad \boldsymbol{g}(\boldsymbol{u}^*, \boldsymbol{v}^*) = \mathbf{0}.$$

Determination of these equilibrium points is the first step in the stability analysis. Depending on the nonlinear structure of ff and gg, there may be a unique equilibrium or several biologically or mathematically admissible equilibria.
The equilibrium is called spatially homogeneous because its value is identical at every point in the domain. Pattern formation concerns the possibility that this homogeneous state loses stability with respect to spatially non-uniform perturbations.
It is useful to distinguish two types of perturbations. A homogeneous perturbation changes the equilibrium by the same amount at every spatial point, whereas a spatial perturbation varies with xx. The reaction system determines the behaviour of homogeneous perturbations, while the full reaction-diffusion system determines the behaviour of both homogeneous and spatially varying perturbations.

2.4 Linearization Around Equilibrium

To investigate local stability, consider a small perturbation of the equilibrium,

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{u}* + \boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{t})$$

where $\boldsymbol{\eta}$ is sufficiently small.
Substitution into the reaction-diffusion equation gives

$$\frac{\boldsymbol{\partial \eta}}{\boldsymbol{\partial t}} = \boldsymbol{D\Delta \eta} + \mathbf{F}(\mathbf{u}^* + \boldsymbol{\eta})$$

Using the Taylor expansion of $\boldsymbol{F}(\boldsymbol{u}^*)$ about $\boldsymbol{u}*$ Ⅎ

$$\mathbf{F}(\mathbf{u}^* + \boldsymbol{\eta}) = \mathbf{F}(\mathbf{u}^*) + \boldsymbol{J\eta} + \boldsymbol{\mathcal{O}}(|\boldsymbol{\eta}|^2)$$

where

$$\boldsymbol{J} = \boldsymbol{D}\text{Ⅎ}\mathbf{b}\boldsymbol{F}(\text{Ⅎ}\mathbf{b}\boldsymbol{u}\text{^}*)$$

is the Jacobian matrix of the reaction kinetics evaluated at the equilibrium.
Because

$$\text{Ⅎb}\boldsymbol{F}(\text{Ⅎb}\boldsymbol{u}\text{^}*) = \boldsymbol{0},$$

the linearized system becomes

$$\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{t}} = \boldsymbol{D}\Delta\boldsymbol{\eta} + \boldsymbol{J}\boldsymbol{\eta}.$$

Neglecting the higher-order nonlinear terms is justified for sufficiently small perturbations when the purpose is to determine local linear stability.

For a two-component system, the Jacobian has the form

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{f_u} & \boldsymbol{f_v} \\ \boldsymbol{g_u} & \boldsymbol{g_v} \end{pmatrix}_{(u^*,v^*)},$$

where

$$\boldsymbol{f_u} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}, \qquad \boldsymbol{f_v} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}, \qquad \boldsymbol{g_u} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}, \qquad \boldsymbol{g_v} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{v}}.$$

The linearized equations therefore become

$$\frac{\partial \boldsymbol{\eta_1}}{\partial \boldsymbol{t}} = \boldsymbol{d_u}\Delta\boldsymbol{\eta_1} + \boldsymbol{f_u}\boldsymbol{\eta_1} + \boldsymbol{f_v}\boldsymbol{\eta_2},$$

$$\frac{\partial \boldsymbol{\eta_2}}{\partial \boldsymbol{t}} = \boldsymbol{d_v}\Delta\boldsymbol{\eta_2} + \boldsymbol{g_u}\boldsymbol{\eta_1} + \boldsymbol{g_v}\boldsymbol{\eta_2}.$$

This system forms the fundamental starting point for the subsequent stability analysis.

2.5 Spatial Mode Decomposition

The effect of diffusion can be analyzed by considering eigenfunctions of the Laplacian. Let $\boldsymbol{\phi_k}(\boldsymbol{x})$ satisfy

$$-\Delta\boldsymbol{\phi_k} = \boldsymbol{\mu_k}\boldsymbol{\phi_k},$$

together with the selected boundary conditions, where

$$\leq \boldsymbol{\mu_0} \leq \boldsymbol{\mu_1} \leq \boldsymbol{\mu_2} \leq$$

are the corresponding Laplacian eigenvalues.

For homogeneous Neumann boundary conditions,

$$\boldsymbol{\mu_0} = \boldsymbol{0}$$

corresponds to the spatially homogeneous mode. The remaining eigenvalues,

$$\boldsymbol{\mu_k} > \boldsymbol{0}, \qquad \boldsymbol{k} \geq \boldsymbol{1},$$

represent spatially varying modes.

A perturbation can be represented as a sum of these spatial modes:

$$\text{Ⅎm}\{\boldsymbol{\eta}\}(\boldsymbol{x},\boldsymbol{t}) = \sum_\boldsymbol{k}\, \text{Ⅎb}\boldsymbol{a}_\boldsymbol{k}(\boldsymbol{t})\boldsymbol{\phi}_\boldsymbol{k}(\boldsymbol{x}).$$

Substituting this expansion into the linearized reaction-diffusion system and using

$$\Delta\boldsymbol{\phi_k} = -\boldsymbol{\mu_k}\boldsymbol{\phi_k}$$

gives

$$\frac{\boldsymbol{da_k}}{\boldsymbol{dt}} = (\boldsymbol{J} - \boldsymbol{\mu_k}\boldsymbol{D})\boldsymbol{a_k}.$$

]Thus, the original partial differential equation is reduced to a sequence of finite-dimensional systems,

$$\text{Ⅎb}\boldsymbol{a}_\boldsymbol{k}'(\boldsymbol{t}) = \boldsymbol{A}_\boldsymbol{k}\text{Ⅎb}\boldsymbol{a}_\boldsymbol{k}(\boldsymbol{t}),$$

where

$$\boldsymbol{A_k} = \boldsymbol{J} - \boldsymbol{\mu_k}\boldsymbol{D}.$$

This relation is fundamental because it shows explicitly how diffusion modifies the stability of each spatial mode. For the homogeneous mode, μ0=0\mu_0=0, and therefore

$$A_0 = J.$$

Consequently, the stability of the homogeneous mode is determined solely by the reaction Jacobian. For non-zero spatial modes,

$$A_k = J - \mu_k D,$$

so the diffusion coefficients directly influence the eigenvalues governing perturbation growth or decay.

2.6 Basic Stability Criteria

Consider first the reaction system without diffusion,

$$\frac{dfu}{dt} = fF(fu).$$

The linearized system around Ⅎ**b** is

$$\frac{d\eta}{dt} = J\eta.$$

The equilibrium is locally asymptotically stable if all eigenvalues of JJ have negative real parts.

For the two-component system, let

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^2 - (J)\lambda + \det(J) = 0,$$

where

$$(J) = a + d$$

and

$$\det(J) = ad - bc.$$

The homogeneous reaction equilibrium is locally asymptotically stable when

$$(J) < 0$$

and

$$\det(J) > 0.$$

These conditions ensure that both eigenvalues have negative real parts.

Once diffusion is introduced, the corresponding matrix for the spatial mode with eigenvalue μ is

$$A(\mu) = J - \mu D.$$

For the diagonal diffusion matrix

$$D = \begin{pmatrix} d_u & 0 \\ 0 & d_v \end{pmatrix},$$

we obtain

$$A(\mu) = \begin{pmatrix} a - d_u\mu & b \\ c & d - d_v\mu \end{pmatrix}.$$

The growth rates of spatial perturbations are therefore determined by the eigenvalues $\lambda(\mu)$ satisfying

$$\det[\lambda I - (J - \mu D)] = 0.$$

The resulting relation between λ and μ is referred to as the **dispersion relation**. Its analysis provides the central mathematical tool for determining whether particular spatial modes are stable or unstable.

For the two-component system, the characteristic equation becomes

$$\lambda^2 - \big(A(\mu)\big)\lambda + \det\big(A(\mu)\big) = 0,$$

where

$$\big(\boldsymbol{A(\mu)}\big) = (\boldsymbol{J}) - (\boldsymbol{d_u} + \boldsymbol{d_v})$$

and

$$\mathbf{det}\big(\boldsymbol{A(\mu)}\big) = \mathbf{det}(\boldsymbol{J}) - (\boldsymbol{ad_v} + \boldsymbol{dd_u})\boldsymbol{\mu} + \boldsymbol{d_u d_v \mu^2}.$$

These expressions show clearly how diffusion changes the stability of the equilibrium. In particular, although the trace becomes more negative when μ increases and diffusion coefficients are positive, the determinant can decrease at an intermediate range of spatial wave numbers. This is one of the mathematical possibilities for diffusion-driven instability. The spatial mode corresponding to μ=0 recovers the stability of the reaction system. For μ>0, however, the diffusion coefficients alter the structure of eigenvalues. A Turing instability occurs when the homogeneous mode is still stable but at least one non-zero spatial mode is unstable. This mode-based approach is the basis of the next section where the dispersion relation is investigated and the conditions for diffusion-driven instability are defined. Striking a balance between stability of the reaction kinetics and stability of the full reaction-diffusion system is important to understand how spatial patterns arise from an initially homogeneous state.

3. Stability Analysis and Diffusion-Driven Instability

The stability of an equilibrium state is a central issue in the mathematical analysis of reaction-diffusion systems. In particular, the formation of spatial patterns is closely related to the possibility that a homogeneous equilibrium, stable under the local reaction dynamics, becomes unstable when diffusion is introduced. This phenomenon, commonly referred to as diffusion-driven instability or Turing instability, provides a fundamental mathematical mechanism for the generation of spatial structure.

Consider the two-component reaction-diffusion system

$$\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial t}} = \boldsymbol{d_u \Delta u} + \boldsymbol{f(u,v)}, \qquad \frac{\boldsymbol{\partial v}}{\boldsymbol{\partial t}} = \boldsymbol{d_v \Delta v} + \boldsymbol{g(u,v)},$$

and let$(\boldsymbol{u^*, v^*})$be a spatially homogeneous equilibrium satisfying

$$\boldsymbol{f(u^*, v^*) = 0}, \qquad \boldsymbol{g(u^*, v^*) = 0}.$$

The stability analysis begins by considering small perturbations of this equilibrium. The reaction terms are linearized about $(\boldsymbol{u^*, v^*})$,resulting in a system governed by the Jacobian matrix

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{f_u} & \boldsymbol{f_v} \\ \boldsymbol{g_u} & \boldsymbol{g_v} \end{pmatrix}_{(\boldsymbol{u^*, v^*})}.$$

For convenience, denote the entries of J by

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}.$$

3.1 Stability of the Reaction System

Before considering diffusion, it is useful to examine the corresponding spatially homogeneous reaction system,

$$\frac{\boldsymbol{du}}{\boldsymbol{dt}} = \boldsymbol{f(u,v)}, \qquad \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \boldsymbol{g(u,v)}.$$

The linear stability of the equilibrium is determined by the eigenvalues of J. For a two-dimensional system, the equilibrium is locally asymptotically stable when

$$\boldsymbol{tr(J) < 0, \det(J) > 0}.$$

3.2 Stability of Spatial Modes

To investigate spatial perturbations, consider the eigenfunctions of the Laplacian associated with the prescribed boundary conditions. Let μk\mu_k denote the corresponding eigenvalues, with

$$\mathbf{0} = \boldsymbol{\mu_0} < \boldsymbol{\mu_1} \leq \boldsymbol{\mu_2} \leq$$

for homogeneous Neumann boundary conditions.

The mode corresponding to $\boldsymbol{\mu_0} = \mathbf{0}$ represents a spatially homogeneous perturbation. The modes with $\boldsymbol{\mu_k} > \mathbf{0}$ represent spatially varying perturbations.

After expanding the perturbation in these spatial modes, the evolution of the amplitude of the k-th mode is governed by

$$\frac{\boldsymbol{da_k}}{\boldsymbol{dt}} = (\boldsymbol{J} - \boldsymbol{\mu_k D})\boldsymbol{a_k},$$

where

$$\boldsymbol{D} = \begin{pmatrix} \boldsymbol{d_u} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{d_v} \end{pmatrix}$$

is the diffusion matrix.

The important feature of this expression is that the stability matrix depends explicitly on the spatial eigenvalue $\boldsymbol{\mu_k}$. Consequently, different spatial modes can have different stability properties even though they originate from the same equilibrium.

For a particular spatial mode with eigenvalue μ\mu, the relevant matrix is

$$\boldsymbol{A(\mu)} = \boldsymbol{J} - \boldsymbol{\mu D}.$$

The temporal growth rate of that mode is determined by the eigenvalues λ\lambda of A(μ)A(\mu). These eigenvalues satisfy the characteristic equation

$$\mathbf{det}[\boldsymbol{\lambda I} - (\boldsymbol{J} - \boldsymbol{\mu D})] = \mathbf{0}.$$

The resulting relationship between λ\lambda and μ\mu is called the **dispersion relation**. It describes how rapidly each spatial mode grows or decays.

3.4 Interpretation of the Dispersion Relation

The dispersion relation provides more than a simple stability criterion. It identifies which spatial scales are amplified most strongly during the initial stage of pattern formation.

Suppose $\boldsymbol{\lambda_+(\mu)}$denotes the eigenvalue having the largest real part. If

$$\boldsymbol{Re\lambda_+(\mu)} < \mathbf{0}$$

for every admissible μ\mu, all small perturbations decay and the equilibrium remains stable.

If there exists an admissible value $\boldsymbol{\mu} = \boldsymbol{\mu_c}$ such that

$$\boldsymbol{Re\lambda_+(\mu_c)} > \mathbf{0},$$

then the corresponding spatial mode grows exponentially during the linear stage of the evolution. The value of μ that creates the largest growth rate is associated with the dominant spatial mode. Its wavelength provides an estimate of the characteristic spatial scale of the emerging pattern. Therefore, the dispersion relation is a direct mathematical relation between stability analysis and pattern choice. It is important to distinguish between the onset of instability and the final nonlinear pattern. Linear stability analysis describes the initial amplification or decay of small perturbations. Once an unstable perturbation gets too large, nonlinear terms become important and determine the amplitude and details of the pattern.

3.5 Role of Differential Diffusion

Diffusion has been intuitively linked with the smoothing of spatial fluctuations. In single-component systems, this interpretation is quite valid, as diffusion always reduces concentration gradients. However, in a coupled reaction-diffusion system, different components can diffuse at different velocity. The ratio between du and dv can strongly influence the stability spectrum. One of the substances can diffuse much faster than the other, changing the balance between the local reaction and transport in space. Accordingly, the diffusion does not contradict the claim that diffusion play a smoothing role. This conclusion is essential for the mathematical framework of Turing patterns. The formation of these patterns happens due to the combination of nonlinear reaction kinetics with different diffusion characteristics. While the reaction processes define local stability, diffusion instead defines the stability of perturbations depending on their position in space.

3.6 Mathematical Significance

This stability framework discussed above, is an excellent tool that systematizes the method of identifying patterns of behaviour. First, the equilibrium is homogeneous: then we have to determine the Jacobian and check the stability. Next, we need to introduce the diffusion matrix, that will lead us to the mode-dependent stability matrix. In this way, we can analyze the eigenvalues of the matrix which will reveal the dispersion relation and identify the spatio-temporal modes that cause the instability. This method is successful, because it transforms the complicated problem of pattern formation into the series of stable problems that can be solved using mathematics. Moreover, this technique gives clear interpretation of the parameters, which influence the instability. In fact, we see that pattern formation can be analyzed as a spectral property that takes place when certain spatial modes are able to gain a positive growth.

All the results obtained here will serve as the basis for the following section, which will give the examples of the reaction-diffusion models, showing that the instability generated by diffusion leads to the formation of real spatial structures. The key point is the connection between the value of parameters, the processes of instability and

4. Conclusion

Reaction-diffusion equations provide a powerful mathematical framework for studying the interactions between local nonlinear dynamics and spatial diffusion. In this work, we designed a mathematical framework to study the stability of homogeneous equilibria and the mechanisms of spatial patterns. We demonstrated that the behaviour of a reaction-diffusion system is determined by the combined effects of reaction kinetics, diffusion coefficients, spatial modes and boundary conditions. Linearization of the reaction terms around a homogeneous equilibrium allowed us to study local stability. In the absence of diffusion, stability is determined by the eigenvalues of the Jacobian matrix associated with the reaction kinetics. When diffusion is incorporated, the problem becomes spatially dependent. This formulation clearly shows how diffusion affects the stability of the individual spatial modes. A key result of the analysis is that diffusion-driven instability is a mathematical mechanism by which a homogeneous equilibrium can become unstable under the influence of spatially heterogeneous perturbations. In particular, a reaction system which is stable locally without diffusion may be unstable for a given range of spatial modes if the interacting components are distributed with the appropriate diffusion rate. The instability is associated with the classical Turing mechanism of pattern formation. In this process, the dispersion relation plays a crucial role as it determines the growth or decay rate of each spatial mode.

Modes with negative growth rates decay, while the ones with positive growth rates increase. The dominant unstable mode thus provides information about the characteristic spatial size of the pattern in the early stage of evolution. Hence stability analysis establishes a mathematical link between the system parameters and the spatial structure of the model. We also highlight differential diffusion. Diffusion alone does not smooth spatial variations; but different diffusion rates between interacting components can change the stability spectrum and destabilize different spatial modes. Therefore, the formation of patterns should not be regarded as a result of diffusion acting independently, but rather as a result of the interaction between reaction kinetics and spatial transport. The mathematical framework presented here is applicable to a wide range of nonlinear reaction-diffusion systems. However, the present analysis is based on linear stability theory and thus describes the beginning and the initial onset of the instability. The size, shape and long-term stability of the pattern are determined by nonlinear effects that need to be studied. Several extensions are very promising for future work. These include nonlinear stability and bifurcation analysis, cross-diffusion systems, non-diagonal diffusion matrices, heterogeneous and nonlocal reaction-diffusion models, time-delayed systems and fractional reaction-diffusion equations. The use of numerical and computational methods would also provide a more detailed theoretical framework to explain the nonlinear evolution and long-term behaviour of spatial patterns. Overall, the mathematical theory of reaction-diffusion equations is a helpful tool to connect equilibrium stability, diffusion-driven instability, spatial modes and pattern formation. The interaction of these mechanisms helps to explain how complex spatial structures are produced out of very simple nonlinear dynamics and helps to lead to more analytic and computational investigation of pattern-forming phenomena.

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