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Analytical Investigation of Magnetohydrodynamic Convective Flow: A Comparative Review of Exact and Approximate Solution Techniques

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1. Introduction

Magnetohydrodynamics (MHD) is an interdisciplinary subject researching the behavior of electric conduction fluids subjected to a magnetic field. The subject combines the principles of fluid mechanics, electromagnetism, and heat transfer for understanding the movement of conductive fluids including the liquid metals, plasma, molten salts, electrolytes, and ionized gases. Following the pioneering work of Hartmann on electrically conducting fluids, magnetohydrodynamic convection has become one of the important fields of study due to its many applications in engineering, industry, geophysics, astrophysics, and energy systems.

Convective transport in MHD has many applications in different industries, for example, in nuclear reactor cooling systems, electromagnetic casting, crystal growth, metallurgical processing, MHD power generation, geothermal energy extraction, electronic device cooling, and biomedical engineering. Thus, the magnetic field gives the opportunity to change the velocity and the thermal boundary layers with the help of the Lorentz force which has influence on fluid stability, heat transfer, and overall efficiency of the system. As a result, it becomes important to study the mathematical behavior of magnetohydrodynamics convective flow in order to optimize industrial processes. The governing equations for magnetohydrodynamic convection are usually made up of coupled nonlinear partial differential equations obtained through mass, momentum and energy conservation principles. The equations become much more intricate when different physical aspects like heat radiation, viscous dissipation, porous medium, chemical reactions, variable fluid characteristics, Hall effect, Joule heating, or buoyancy-driven convection are added to the equations. In fact, it is already very difficult to find exact analytical solutions of nonlinear equations in most real scenarios, which justifies the use of various sorts of analytical and semi-analytical methods.

A number of approaches to derive the mathematical solutions of MHD convective flows problems have been developed in the last decades. Classical techniques, including the method of separating variables, similarity transformation, Laplace transformations, Fourier transformations, and integral transformations, have offered elegant exact solutions for the simplified problems under quite stringent assumptions. However, in most cases, real-life problems of MHD involve nonlinearities and complex boundary conditions unsuitable for the use of the aforementioned classical methods. This has resulted in the

development of some approximate analytical methods like perturbation techniques, method of homotopy analysis (HAM), method of homotopy perturbation approaches (HPM), method of Adomian decomposition (ADM), method of varying iteration (VIM), method of differential transformation (DTM), and method of Lie symmetry. With the help of these approaches those problems of MHD can be solved.
There are important differences between the mathematical characteristics, assumptions, convergence features, computational requirements, and applicability domains of the different analytical methods. Exact methods often lead to obtaining closed-form solutions that give straightforward physical interpretations; however, they can only be applied to simplified geometries and ideal boundary conditions. On the other hand, approximate analytical techniques allow solving very nonlinear problems and complicated physical models but their convergence, stability, and accuracy often depend on the assumptions and parameters dictated by the particular problems at hand. This makes the process of determining the analytical approach very complicated for scientists dealing with magnetohydrodynamic convection.
Even though numerous review papers deal with the description of different analytical methods and particular applications of MHD flow problems, comparative studies that classify analytical methods and analyze their strengths, disadvantages, and applicability are rare. The analyses provided in the studies mainly concentrate on numerical simulations and specific physical models and do not offer a general mathematical view of analytical approaches to solving the problems. Thus, a qualitative comparison of the existing analytical methods would be useful for choosing the best techniques for different types of MHD convection problems and for determining the future trends in the theoretical studies. The primary objective of the paper is to carry out a thorough analytical assessment of the methods used to solve MHD convection issues. The focus of this paper is not on the computational aspects of the methods, but rather on the theory behind them. The classical and new approximate methods are classified and compared with respect to their mathematical representations, convergence criteria, complexity, suitability for solving nonlinear problems, and ability to solve real engineering problems. Moreover, in this paper, the current trends and possible future directions of MHD convection modelling are outlined.

The specific objectives of the paper are as follows:
1. To propose the classification of exact, semi-analytical, and symmetry-based techniques for solving MHD problems.
2. To compare the most commonly used analytical methods according to their theoretical background, underlying assumptions, convergence results, pros, and cons.
3. To provide practical recommendations on the selection of the most appropriate analytical technique depending on the specific features of a given problem.
4. To identify the existing gaps in the research and suggest possible future investigations.

It is anticipated that this comparative study will serve as a useful reference for mathematicians, physicists, and engineers engaged in theoretical and applied research on magnetohydrodynamic convection, while also providing a foundation for future developments in analytical modeling of complex transport phenomena.

2. Mathematical Formulation of Magnetohydrodynamic Convective Flow

Magnetohydrodynamic (MHD) convection flow refers to the movement of a fluid that conducts electricity in the presence of heating and an external magnetic field. This problem is formulated mathematically by applying the main conservation laws in fluid dynamics according to Maxwell's equations of electromagnetic field theory. Depending on the particular mechanical configuration and application, the governing equations can become more complicated by including thermal radiation, viscosity, chemical reactions, Joule heating, porous resistance, Hall currents, and temperature-dependent thermophysical properties into the model. However, the classical MHD flow theory is often accepted as the basis for developing different models.

2.1 Basic Assumptions

For the purpose of deriving the governing equations, the authors usually accept several essential assumptions:

- The fluid is incompressible and a good conductor of electricity;
- The flow is laminar and Newtonian;
- Fluid properties are constant, except for the density changes, which are responsible for buoyancy (Boussinesq approximation);
- The induced magnetic field is negligible in comparison with the applied magnetic field, due to a low magnetic Reynolds number;
- There is no external electric field;
- The viscous forces and electromagnetic forces act together on the fluid;
- Heat is transferred by conduction and convection.

These assumptions allow us to develop a simpler

2.2 Governing Equations

The mathematical description of MHD convection consists of the continuity equation, momentum equation, and energy equation.

Continuity Equation

For an incompressible fluid, conservation of mass requires

$$\nabla \cdot V = 0,$$

where

$\big(V = (u, v, w)\big)$ denotes the velocity vector.

This equation ensures that the fluid density remains constant throughout the flow field.

Momentum Equation

The momentum equation for an incompressible electrically conducting fluid may be written as

$$\rho\left(\frac{\partial V}{\partial t} + (V \cdot \nabla)V\right) = -\nabla p + \mu\nabla^{2V} + \rho g\beta(T - T_\infty) + J \times B$$

where

- ρ is the fluid density,
- p is the pressure,
- μ is the dynamic viscosity,
- g denotes gravitational acceleration,
- β represents the thermal expansion coefficient,
- T is the fluid temperature,
- T_∞ is the ambient temperature,

- J is the electric current density,
- B is the magnetic field intensity.

The final term,

$$J \times B,$$

represents the Lorentz force, which distinguishes magnetohydrodynamic flows from ordinary convective flows. This force generally opposes the fluid motion and alters the velocity distribution within the boundary layer.

Ohm's Law

Under the assumption of negligible electric field,

$$\boldsymbol{J}$$
$$\sigma(V \times B),$$

where

- (σ) denotes the electrical conductivity.

Consequently,

$$\boldsymbol{J} \times \boldsymbol{B}$$
$$-\sigma B_0^{2V},$$

for a uniform transverse magnetic field (B_0).

Substituting this relation into the momentum equation introduces magnetic damping through the Lorentz force.

Energy Equation

The conservation of thermal energy is expressed as

$$\boldsymbol{\rho c_p}\left(\frac{\boldsymbol{\partial T}}{\boldsymbol{\partial t}} + \boldsymbol{V} \cdot \boldsymbol{\nabla T}\right) k\nabla^2 T + Q,$$

where

- (c_p) denotes the specific heat,
- (k) is the thermal conductivity,
- (Q) represents internal heat generation or absorption.

For more advanced models, additional terms corresponding to viscous dissipation, thermal radiation, and Joule heating may be incorporated.

2.3 Dimensionless Formulation

To facilitate mathematical analysis, the governing equations are usually transformed into dimensionless form by introducing appropriate characteristic scales for velocity, length, temperature, and pressure.

Typical dimensionless variables are defined as

$$X = \frac{x}{L},$$

$$Y = \frac{y}{L},$$

$$U = \frac{u}{U_0},$$

$$V = \frac{v}{U_0},$$
$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

where

- (L) is the characteristic length,
- (U_0) is the reference velocity,
- (T_w) is the wall temperature.

Using these transformations leads to a set of dimensionless governing equations containing several important physical parameters.

2.4 Important Dimensionless Parameters

The behavior of MHD convective flow is primarily governed by the following dimensionless numbers.

Reynolds Number

$$Re = \frac{\rho U_0 L}{\mu},$$

which measures the ratio of inertial forces to viscous forces.

Grashof Number

$$Gr = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2}$$

representing the relative importance of buoyancy forces compared with viscous forces.

Prandtl Number

$$\boldsymbol{Pr} = \frac{\nu}{\alpha},$$

which characterizes the ratio of momentum diffusivity to thermal diffusivity.

Hartmann Number

$$\boldsymbol{Ha} = B_0 L \sqrt{\frac{\sigma}{\mu}},$$

measuring the strength of the applied magnetic field relative to viscous effects.

Magnetic Interaction Parameter

$$\boldsymbol{M} = \frac{\sigma B_0^2 L}{\rho U_0},$$

that measures the effect of magnetic fields on the movement of the fluid. High amounts of M usually reduce the velocity of fluid because of high forces due to Lorentz effect.

2.5 Similarity Transformations

Many boundary layer MHD problems can be simplified by using similarity variables which turn the equations from partial differential to ordinary differential.
One of the most commonly used similarity variables is

$$\boldsymbol{\eta} = y \sqrt{\frac{U_0}{\nu x}},$$

together with the stream function

$$\boldsymbol{\psi} = \sqrt{\nu U_0 x}, f(\eta),$$

where

$$\boldsymbol{u} = \frac{\boldsymbol{\partial\psi}}{\boldsymbol{\partial y}},$$

$$\boldsymbol{v} = -\frac{\partial\psi}{\partial x}.$$

The mathematical transformations lessen the complexity of governing equations and provide the foundation for numerous analytical solution methods discussed in the subsequent sections.

2.6 Importance of the Presented Mathematical Model

The mathematical formulation mentioned above can be considered a theoretical foundation of almost all analytic studies of magnetohydrodynamic convective flow. Depending on the physical problem being studied, it is possible to modify governing equations by adding extra transportation mechanisms like porous media resistance, thermal radiation, chemical reactions, the effects of the nanofluid, non-

Newtonian rheology, etc. However, the basic conservation laws will still be valid, while the major aim of most analytical solution techniques is to find either exact or approximate solutions to these coupled nonlinear equations. Hence, a clear understanding of the mathematical form of the governing equations must precede the consideration of the different analytical methods applied in the field of MHD.

3. Types of Exact and Approximate Solution Methods

The nonlinear character of the governing equations of magnetohydrodynamic (MHD) convective flow requires the use of different analytical solutions methods. Generally speaking, governing system can be written in the following form

$$\mathcal{L}(u) + \mathcal{N}(u) = f,$$

where $(\mathcal{L})$ denotes the linear operator, $(\mathcal{N})$ represents the nonlinear operator, and $(f)$is the source term. The choice of an analytical method depends on the characteristics of $(\mathcal{N})$, boundary conditions, and the desired solution accuracy.

3.1 Proposed Classification Framework

Exact Analytical Methods

- Separation of Variables
- Similarity Transformations
- Laplace Transform Method
- Fourier Transform Method
- Integral Transform Techniques

Semi-Analytical Methods

- Regular and Singular Perturbation Methods
- Homotopy Analysis Method (HAM)
- Homotopy Perturbation Method (HPM)
- Adomian Decomposition Method (ADM)
- Variational Iteration Method (VIM)
- Differential Transform Method (DTM)

Symmetry-Based Methods

- Lie Symmetry Analysis
- Scaling Transformations
- Similarity Reduction Techniques

This classification shows a transition of traditional approaches to techniques, which are more advanced.

3.2 Exact Analytical Methods

Exact methods are able to give solutions in a closed form under some specific assumptions regarding properties of media that are usually based on factors that are said to be stable – constant or not changing, linear or being represented by simple shapes.

$$\boldsymbol{\eta = y\sqrt{\frac{U_\infty}{\nu x}}},$$

which convert the governing partial differential equations into ordinary differential equations,

$$F(f, f', f'', \ldots) = 0.$$

Integral transforms are also frequently employed. For example, the Laplace transform is defined by

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t), dt,$$

which turns differential equations into algebraic equations that are simpler to solve.
While exact methods yield benchmark solutions with understandable physical significance, they are often limited to simplified MHD models.

3.3 Semi-Analytical (Approximate) Methods

When there are no exact solutions for nonlinear MHD problems, semi-analytical methods yield convergent analytical approximations while preserving the structure of the governing equations.

Perturbation Methods

The solution is expressed in the form of

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots,$$

where (ε) is a small perturbation parameter. These methods are accurate only when ($\varepsilon \ll 1$).

Homotopy Analysis Method (HAM)

HAM constructs the zeroth-order deformation equation

$$(1 - p)\mathcal{L}[\phi - u_0] = ph\,\mathcal{N}[\phi]$$

where ($p \in [0,1]$) is the embedding parameter and (h) is the convergence-control parameter.

Homotopy Perturbation Method (HPM)

HPM expresses the solution as

$$u = \sum_{n=0}^{\infty} p^n u_n,$$

combining perturbation theory with homotopy concepts to obtain rapidly convergent series solutions.

Adomian Decomposition Method (ADM)

ADM represents the solution by

$$u = \sum_{n=0}^{\infty} u_n ,$$

while nonlinear terms are decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n ,$$

where A_n are Adomian polynomials.

Variational Iteration Method (VIM)

The iterative correction formula is

$u_{n+1} = u_n + \int \lambda(\xi) R(u_n) \,, d,$

where $R(u_n)$ denotes the residual operator and (\lambda) is the Lagrange multiplier.

Differential Transform Method (DTM)

DTM converts differential equations into recursive algebraic relations using

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right] x = 0,$$

and reconstructs the solution through

$$u(x) = \sum_{k=0}^{\infty} U(k) x^k .$$

Such techniques have become widespread in applications related to non-linear MHD convection since they circumvent discretization while giving analytic approximations.

3.4 Symmetry-based Techniques

Symmetry-based techniques allow for a simplification of non-linear differential equations through invariant transformations. The one-parameter Lie transformation can be written as follows:

$$\boldsymbol{x}^{=}\boldsymbol{x} + \boldsymbol{\varepsilon\xi}(\boldsymbol{x}, \boldsymbol{u}),$$
$$\boldsymbol{u}^{=}\boldsymbol{u} + \boldsymbol{\varepsilon\phi}(\boldsymbol{x}, \boldsymbol{u}),$$

where (ε) is an infinitesimal parameter.

The corresponding infinitesimal generator is

$$\boldsymbol{X} = \boldsymbol{\xi}(\boldsymbol{x}, \boldsymbol{u}) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{u}) \frac{\boldsymbol{\partial}}{\boldsymbol{\partial u}} .$$

These transformations reduce governing PDEs to ODEs through similarity variables, making subsequent analytical treatment significantly easier. Symmetry methods are particularly useful for nonlinear MHD boundary-layer problems.

3.5 Comparative Characteristics

The mathematical philosophy of the three classes is fundamentally different. On the one hand, exact methods yield benchmark solutions in closed forms, but they come with a lot of constraints. On the other hand, semi-analytical methods allow for their application not only to linear systems but to nonlinear systems through series of convergent expansion. The last category is represented by symmetry methods, used for managing mathematical complexity ahead of solving equations.
No analytical method can be the best for all applications, but it may be appropriate for the particular problem depending on its governing equations, level of nonlinearity, boundary conditions as well as the convergence requirements.
The selection of the analytical method can be summarized in the following way:
Exact methods: appropriate for linear or weakly nonlinear equations defined by simple geometries. Semi-analytical methods: applicable to highly nonlinear problems requiring approximate analytical results. Symmetry methods: suitable for the problems which are reduced by similarity limiting or reduced by invariant transformation.
The approaches listed above may be used in conjunction in many cases. For instance, similarity transformations reduce the system of PDEs to the system of ODEs that would be solved through HAM, ADM, or VIM yielding analytical convergent solution.

4. Comparative Analysis of Exact and Approximate Solution Techniques

The governing equations of MHD convective flow are highly nonlinear and coupled, making their analytical solution challenging. In general, the mathematical model can be represented as

$$\boldsymbol{\mathcal{L}}(\boldsymbol{u}) + \boldsymbol{\mathcal{N}}(\boldsymbol{u}) = \boldsymbol{f},$$

where (\mathcal{L}) is the linear operator, (\mathcal{N}) the nonlinear operator, and (f) the forcing function. Different analytical techniques solve this equation by exploiting linearity, perturbation, decomposition, variational principles, or symmetry transformations.

4.1 Exact Analytical Methods

Exact methods provide closed-form solutions under simplifying assumptions such as linearity, constant fluid properties, or simple geometries. They commonly employ similarity transformations or integral transforms. For example,

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}},$$

reduces the governing PDEs to ODEs,

$$F(f, f', f'', \dots) = 0.$$

Similarly, the Laplace transform,

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t), dt,$$

converts differential equations into algebraic equations. Exact solutions serve as benchmark solutions but are generally restricted to idealized MHD models.

4.2 Perturbation Methods

Perturbation methods assume the existence of a small parameter (\varepsilon), and the solution is expressed as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots.$$

Successive coefficients are determined by equating equal powers of (\varepsilon). Although highly accurate for weak nonlinearities, these methods become ineffective when no suitable perturbation parameter exists.

4.3 Homotopy Analysis Method (HAM)

HAM constructs a continuous deformation between an initial approximation and the exact solution through

$$(1-p)\,\mathcal{L}[\phi(\eta;p) - u_0(\eta)] = ph\,H(\eta)\,\mathcal{N}[\phi(\eta;p)]$$

where $p \in [0,1]$is the embedding parameter and (h) controls convergence. Unlike perturbation methods, HAM does not require small parameters and is well suited for strongly nonlinear MHD problems.

4.4 Homotopy Perturbation Method (HPM)

HPM combines homotopy theory with perturbation concepts by expressing the solution as

$$u = \sum_{n=0}^{\infty} p^n u_n,$$

where (p) is an embedding parameter. The method provides rapidly convergent analytical approximations with relatively simple mathematical implementation, although convergence may deteriorate for highly nonlinear systems.

4.5 Adomian Decomposition Method (ADM)

ADM decomposes the solution and nonlinear operator into infinite series,

$$u = \sum_{n=0}^{\infty} u_n ,$$
$$N(u) = \sum_{n=0}^{\infty} A_n ,$$

where A_n are Adomian polynomials representing nonlinear terms. The method avoids linearization and discretization, making it suitable for a wide class of nonlinear MHD equations.

4.6 Variational Iteration Method (VIM)

VIM constructs successive approximations using the correction functional

$$u_{n+1} = u_n + \int \lambda(\xi) , R(u_n), d\xi,$$

where $R(u_n)$ denotes the residual and (\lambda(\xi)) is the Lagrange multiplier. The method converges rapidly for many nonlinear boundary-layer problems without requiring mesh generation.

4.7 Differential Transform Method (DTM)

DTM transforms differential equations into recursive algebraic equations through

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0} ,$$

and reconstructs the solution using

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k .$$

DTM is computationally efficient for local analytical solutions but has a limited radius of convergence.

4.8 Lie Symmetry Analysis

Lie symmetry analysis exploits invariance under continuous transformation groups. The infinitesimal transformation is given by

$$x^{=}x + \varepsilon\xi(x,u),$$
$$u^{=}u + \varepsilon\phi(x,u),$$

with generator

$$X = \xi(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u}.$$

These symmetries reduce PDEs to ODEs through similarity variables, providing an effective analytical framework for nonlinear MHD models.

4.9 Comparative Assessment

Different analytic techniques have particular strengths determined by the particular features of their belonging equations. Exact analytical methods yield reliable results but require many constraints in assumptions. Perturbation techniques can be used only when there is a good perturbation parameter. HAM is superior as for convergence control with strongly nonlinear equations, while HPM has easier implementation than the others. ADM and VIM do not use discretization and can deal with nonlinear operators, while DTM is great for the local series case. Lie symmetry analysis is combined with these techniques as it minimizes equation complexity before applying any analytical methods.

In summary, it is possible to conclude that the choice of an analytical approach depends on governing equations, non-linearity degree, convergence requirements, and physical assumptions. Similarity transformations are used first in most of the MHD studies to reduce governing PDEs followed by HAM, ADM, and VIM in order to get converging analytical solutions.

The data comparison indicates that modern semi-analytic methods significantly expand the range of MHD convection problems that can be solved analytically. However, classical exact methods still provide important benchmark solutions while symmetry-based approach has proven to be an effective tool for simplifying complex governing equations. The complementary nature of these methods shows that selecting a proper analytical method is crucial in terms of peculiarities of the physical event being investigated.

5. Emerging Trends, Research Gaps, and Future Directions

The swift progress made in the field of magnetohydrodynamic (MHD) research has opened up the field of analytical studies beyond the classical problems of flow in boundary layers. Present-day applications utilize integrated multiphysics processes along with nonlinear constitutive laws and complex transportation processes. Thus, it is reasonable to consider and analyze the current gaps in research and new tendencies in analytical MHD.

5.1 New Trends in Analytical MHD Research

Due to the recent research discoveries made in the field of analysis, many researchers have switched from easy Newtonian models to more complicated systems. There are some noticeable trends in the field of research.

Fractional-Order MHD Models

Fractional calculus has proved its efficacy as a mathematical tool used for the study of memory and hereditary effects in electrically conducting fluids. Fractional derivatives provide more possibilities compared to traditional models with integer order in doing such steps as nonlocal transport and anomalous diffusion. Analysts have used various techniques such as the Method of Homotopy Analysis, the Method of Decomposition and Variational Iteration Method to apply analytical methods to solve fractional MHD equations.

Non-Newtonian Fluid Models

There are many kinds of fluid that demonstrate non-Newtonian behavior, for example, viscoelastic fluids, micropolar fluids, Casson fluids, Maxwell fluids, Jeffrey fluids, and Williamson fluids. Nonlinear constitutive laws of these fluids have complicated things mathematically speaking, which has inspired scientists to resort to new methodologies of analytical approximations.

Hybrid Nanofluids

Hybrid nanofluids have become a topic of a great scope of research because they are attractive because of their high thermal conductivity and high heat transfer effectiveness. Analysis of hybrid nanofluids is increasingly devoted to the study of the effect of many nanoparticles, influence of the magnetic field, and thermal radiation on the processes of convective transport.

Porous Media and Multiphysics Coupling

Most engineering applications involve flow of various fluids through porous structures, as well as heat transfer, mass transport, chemical reactions, and electromagnetic processes.

5.2 Existing Research Gaps

While significant progress has been made in the field, substantial gaps in the research on MHD convective flow still exist.

Lack of Comparative Studies.

Most of the studies focus on the usage of one analytical method for the given flow problem. There are few published works that have compared many analytical methods through the same mathematical criterion, which means that researchers do not have much guidance in terms of the process of choosing the most appropriate method of solution.

Lack of Standard Criterion of Evaluation.

Different studies evaluate the quality of the solution based on their own convergence criteria, which complicates the process of comparison. It would be beneficial to have a unified framework for convergence assessment, efficiency of computations, stability, and accuracy of the analysis.

Complicated Physical Models.

Today's MHD problems often entail the simultaneous use of thermal radiation, porous media, chemical reactions, phase transitions, variable properties of liquids, and many kinds of non-linear boundary conditions. It is very complicated to be able to come up with analytical methods that will be able to solve such difficult problems.

5.3 Future Research Directions

Magnetohydrodynamic future analytical investigations should emphasize several promising research fields. The first aspect of concern is the development of common analytical frameworks. Such frameworks should be complemented by multiple physical factors within one mathematical construction. The use of these frameworks will increase the effectiveness of analytical techniques for real engineering tasks. Hybrid analytical techniques can also be helpful in improving the convergence and accuracy of the solutions due to combined application of similarity transformations and modern semi-analytical approaches. The hybrid method considers disadvantages of each method used while keeping the analytic nature of mathematics. The new analytical methods need to provide rigorous convergence and stability analysis. The mathematical proof of the convergence regions, the unique solution existence and the error reliable information would make the approximate analytical solutions more trustworthy.

Analytical investigations can also include interdisciplinary areas of application such as renewable energy systems, biomedical engineering, advanced manufacturing and environmental fluid mechanics. These fields give new opportunities for the development of analytical methods. Finally, there is a need for comparative studies of various analytical methods. The generation of uniform evaluation criteria will allow assessing the efficiency of the analytical techniques objectively.

5.4 Recommendations

Given the current comparative analysis, the subsequent recommendations have been developed:

1. The application of precise analysis is recommended whenever simplified mathematics allows finding closed-form solutions.
2. Semi-analytic techniques should be used instead of precise methods in the case when the exact solution is impossible to obtain.
3. More active attention should be paid to methods based on symmetry as they allow simplifying the equations.
4. In the future, more emphasis should be placed on the correctness of the convergence analysis in addition to finding solutions.
5. More comparative studies of various analytical methods should be carried out in the future to establish some benchmarks and best practices.

The development of analytical methods is crucial for better understanding the theoretical trends in magnetohydrodynamic convection and for future engineering innovations.

6. Conclusions

Researchin the area of magnetohydrodynamic convective flow is a demanding subject in applied mathematics, fluid mechanics, and engineering. The field is very feasible in various industries, energy systems, metallurgy, geophysics, and heat transfer analyses. However, it presents an interesting challenge for researchers due to the complexity of the problem stemming from the nonlinear interrelation of momentum, heat flow, and electromagnetic effects.

In this regard, this paper presented a broad analytical paper analyzing and comparing the most widely used techniques for solving MHD convective flow problems in practice. A unique classification system was proposed whereby all the analytical methods were grouped into three main groups: exact analytic methods, semi-analytic methods, and symmetry methods. This approach offers a wider view on analytical methods and it simplifies the process of choosing the right method for different types of problems.

It has been shown that exact analytical methods are still key for the analysis of simplified linear models because they provide one-step solutions to the problems along with practical insights into the processes.

But their use is usually constrained by unreasonable mathematical conditions. Semi-analytical approaches like the Homotopy Analysis Method, Homotopy Perturbation Method, Adomian Decomposition Method, Variational Iteration Method, and Differential Transform Method have considerably widened the applicability of nonlinear MHD cases while retaining the structure of various analytical solutions. Symmetry-based approaches such as Lie symmetry analysis remain perfect mathematical instruments for simplifying the governing equations and getting similarity solutions.

A main finding that this research has brought is that there is no method that can be called the best one in general. The choice of methods needs to be based on the mathematical nature of the equations and the level of nonlinearity. The framework developed in this paper presents practical help to scientists working in the sphere of magnetohydrodynamic convection.

Several recent trends in research have been enumerated in this review. These include fractional MHD models, non-Newtonian fluid mechanics, hybrid nanofluids, porous medium transport and multiphysics. The mentioned trends generate new problems that require more advanced mathematical methods, a rigorous convergence analysis and proper assessment criteria to solve their problems effectively. In the case of necessity, combining the use of symbolic computation with classical analytical techniques will allow to improve the accuracy of future analytical studies even more.

As a conclusion, it should be noted that analytical techniques will continue to play an essential role in the understanding of magnetohydrodynamic convective flow. Giving an overview of the state-of-the-art of the existing approaches, identifying the most promising fields for further research, and leading to better understanding of the principles of MHD theory, the paper contributes to the field of study, promoting the search for proper analytical techniques.

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