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# A Critical Study on Fuzzy Hausdorff Space and Its Impact on Topology

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**Abstract**

**Our main aim is to define the relation between the fuzzy set and topological spaces. However the fuzzy sets hypothesis goes in a developing phase, now a days it is accepted spacious and precious applications. We also develop the generalised structure which includes several fields such as point set topology, algebraic topology, and differential topology. Georg Cantor, David Hilbert, Felix Hausdorff, Maurice Frechet and Henri Poincare studied its basic properties based on Euclidean space and some geometrical structure. For that purpose, we have tried to offer more fundamental description of these two topics. In the fuzzy topological spaces section. We derive definition and also proved of theorems. The existing consequences in this analysis signify that large numbers of the fundamental concepts from general topology might be expanded enthusiastically to fuzzy topological spaces.**

**Keywords: Algebraic Topology, Euclidean space, Fuzzy set, Fuzzy Topology and Hausdorff space**

## Introduction

Topology is a special kind of geometry, a geometry that doesn't include a notion of distance. Topology has many roots in graph theory. A topological group is a set that has both a topological structure and an algebraic structure.

We consider a metric space is a generalization of a Euclidean space and a topological space is a generalization of a metric space. Instead of having a metric that tells us the distance between two points, topological spaces rely on a different notion of closeness; points are related by open sets.

### (i) Definition (Arbitrary unions)

Let X be a set. A topology on X is a collection τ of subsets of X that satisfy the following three requirements:

i) $\emptyset \in \tau$ and $X \in \tau$
ii) Given $U \subset \tau$, we have $\cup\{U : U \in U\} \in \tau$ (Closure under arbitrary unions)
iii) Given U1 and U2 $\in \tau$, we have $U1 \cap U2 \in \tau$ (Closure under finite intersections) Members of a topology are called open sets.

**(ii) Definition (Hausdorff)**

A topological space$(X, \tau)$ is called a Hausdorffspace when distinct points can be separated by open sets. In symbols: $\forall x, y \in X$ with $x \neq y$ $\exists U, V \in \tau$ with $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

**(iii) Definition (continuous)**

Let $(X, \tau X)$ and $(Y, \tau Y)$be topological spaces and let $f : X \rightarrow Y$ be a function f is continuous at $x0 \in X$ as follows:

$N \in Nf(x0)$implies $f-1(N) \in Nx0$.

**(iv) Definition (coarser)**

Let X be a set and let $\tau 1$ and $\tau 2$ be topologies on X. If $\tau 1 \subset \tau 2$, then $\tau 1$ are coarser than $\tau 2$.If $\tau 2 \subset \tau 1$, then $\tau 1$ is finer than $\tau 2$.

**(v) Definition (Euclidean topology)**

The elements of the Euclidean topology on Rn are unions of open balls in Rn. This topology is represented by $\|\cdot\|n$. A different case that we will make persistent reference to is R, here open sets are unions of open intervals.

**(vi) Definition (topological space)**

A set X organized with a topology $\tau$ on X forms a topological space. This is represented by the pair$(X, \tau)$.

**(vii) Definition (neighbourhood)**

Let$(X, \tau)$ be a topological space and $x \in X$. A set $N \subset X$ is a neighbourhood of x if there exists some $U \in \tau$ with $x \in U \subset N$. Now, N is a set that holds an open set containing x. The set of all neighbourhoods of an element x will be represented by Nx. Let $(X, \tau)$ be a topological space. We define that a subset F of X is closed when $X \setminus F \in \tau$. https://dcprequirement.in/admit-card-clerk-login-wtest.aspx

Given a topological space $(X, \tau)$, subsets of X can be: open, closed, both open and closed, or neither open nor closed. We contemplate sets that are both open and closed clopen. In the discrete space, every subset of X is a clopen set.

**(viii) Definition (closure)**

Let (X, τ) be a topological space and S a subset of X. The closure of S, denoted $\overline{S}$, is defined to be $\overline{S}$= ∩{F : F ⊂X is closed and contains S}. The closure is inscribed as $\overline{S}$ = {x ∈X :N ∩ S ≠∅ for all N ∈Nx . These two sets are not the equal, in given $\overline{S}$may take as the smallest closed set that holds S.

**(ix) Definition (dense)**

Let (X, τ) be a topological space. We say that a set D ⊂X is dense in X when $\overline{D}$ = X.

**(x) Definition (base)**

Let (X, τ) be a topological space. A base B for τ is a subset of τ where each open set in τ can be written as unions of elements in B.

**(xi) Definition (sub -base)**

Let (X, τ) be a topological space. A sub-base for τ is a set S ⊂τ with the property that the set of all finite intersections of sets in S is a base for τ. Upon initial seeing the definition for a base and subbase it may be tough to understand exactly what is being said, since the two definitions are so similar. The following example should explain the distinction between the two.

**(xii) Definition (fundamental system of neighbourhoods)**

A fundamental system of neighbourhoods may be assumed of as localizing the notion of a base around a sole point. Let (X, τ) be a topological space, let x ∈X, and let Ux be the set of all open neighbourhoods of x. Let Vx be a subset of Ux. We denote that Vx is a essential system of neighbourhoods of x when for all Ux in Ux there exists some Vx in Vx with Vx⊂Ux. We say Vx is a base for Ux.

**Theorem**

Let(X, τX) and(Y, τY)be a topological spaces and let f : X → Y be a function. The following three statements are as follows.

i) f : X → Y is continuous on X,

ii) f−1(U) is open in X for all open sets U in Y,

iii) f−1(F) is closed in X for all closed sets F of Y.

Many declarations that are true for continuous functions in metric spaces are also factual in topological spaces. An illustration of this is the transitivity of continuity.

**Theorem**

Let$(X, \tau X)$ and$(Y, \tau Y)$ and$(Z, \tau Z)$ be topological spaces such that $g : X \to Y$ is continuous at $x0 \in X$ and $f : Y \to Z$ is continuous at $g(x0) \in Y$ . It follows that$(f \circ g)(x)$ is continuous at x0.

**Proof**

By using definition, we have: $g: X\to Y$ is continuous at x0: $N \in Ng(x0)$ implies $g-1(N) \in Nx0$ $f : Y\to Z$ is continuous at $g(x0)$: $M \in Nf(g(x0))$ implies $f-1(M) \in Ng(x0)$. In order to prove that $(f \circ g)(x)$ is continuous at x0 it necessity be shown that $N \in N(f \circ g)(x0)$ implies $g-1(f-1(N)) \in Nx0$. Let us now assume that $N \in N(f(g(x0)))$. Since f is continuous at g(x0), it follows that $f-1(N) \in Ng(x0)$. Since g is continuous at x0 we got $g-1(f-1(N)) \in Nx0$, as essential. Hence, it is proved.

**(i) Definition (directed)**

An ordered set X is called directed when $\alpha, \beta \in X$ implies there exists $\gamma \in X$ with $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \gamma$ .

**(ii) Definition (net)**

A net$(x\alpha)\alpha\in A$ in a set X is a function from a directed set A to X. A net is a generalization of a sequence. We can take subnets of a net in closely the similar way we take sub-sequences of a given sequence.

**(iii) Definition (converges)**

Let$(X, \tau)$ be a topological space. We say that a net$(x\alpha)\alpha\in A \in X$ converges to $x \in X$ when for all $N \in Nx$, there is an $\alpha N \in A$ with $x\alpha \in N$ for all $\alpha \in A$ such that$\alpha N \preccurlyeq \alpha$. This is denoted by $x\alpha \to x$ (x is the limit of the net). The closure of a subset of X in terms of closed sets and neighbourhoods. There is a different formulation of that notion we may build up with nets.

**Theorem**

Let$(X, \tau)$ be a topological space and S a subset of X. The closure of S is the set of all facts in X that are a limit of a net in S. This may be worded as: $x \in \overline{S}$ if and only if x is the limit of a net in S. This is a property of limits in metric spaces that carries over to topological spaces.

**Proof**

Let us consider Nxas a directed set defined by $M \preccurlyeq N$ when $N \subset M$ for $M, N \in Nx$ for some x. Assume that $x \in \overline{S}$, by definition (iii) we find that for all $N \in Nx$, there exists an $xN \in N \cap S$. So, find that the net $(xN)N\in Nx$ converges to x, this is because all neighbourhoods of x contain x. For the other direction let $(x\alpha)\alpha\in A$ be a net in S that converges to x. Let us assume by way of contradiction that $x \notin \overline{S}$, so $x \in X \setminus \overline{S}$. We know that $\overline{S}$ is closed, this implies that $X \setminus \overline{S}$ is open, which means that $X \setminus \overline{S}$ is a neighbourhood of x. Since $(x\alpha)\alpha\in A$ converges to a point outside of $\overline{S}$, there exists some $\alpha N \in A$ such that $x\alpha \in X \setminus \overline{S}$ for all

$\alpha N \preccurlyeq \alpha$. We identify that $x \setminus \overline{S} \subset x \setminus S$, so $x\alpha \in X \setminus S$ for each $\alpha N \preccurlyeq \alpha$. This is a contradiction since $(x\alpha)\alpha \in A$ is a net in S. Hence, the theorem is proved.

**Corollary**

$F \subset X$ is closed if and only if every net in F that converges in X has a limit in F. In metric spaces the limit of a sequence is unique. This is a property that does not hold in topological spaces.

**Proposition**

A topological space$(X, \tau)$ is Hausdorff if and only if every convergent net in the space has a unique limit.

**(i) Definition (topological space)**

Let$(X, \tau)$ be a topological space and let $Y \subset X$. We define$\tau|Y = \{Y \cap U : U \in \tau\}$. Hence, this set is a topology on Y. which is said to be topology inherited from X.

**(ii) Definition (coordinate projections)**

Let $((Xi, \tau i))i \in I$be a family of topological spaces and $(X, \tau)$ be the topological product. The functions $\pi i : X \rightarrow Xi$ defined by $\pi i$ $((x1, x2, ...xi, ....)) = xi$ are said to the coordinate projections.

**(iii) Definition (topological space)**

Let$((Xi, \tau i))i \in I$be a family of topological spaces and let $X = \epsilon i \in IXi$ as sets. It depends on whether or not I is a finite set two different topologies on X have been defined. If I is finite, then we use the box topology:$\tau = \{\Pi i \in IUi : Ui \in Ti.\}$. If I is an infinite set then the box topology too different elements to retain its desirable properties (Tychonoff's Theorem doesn't hold). we now define $\tau$ to be the coarsest topology that makes the coordinate projections $\pi i : X \rightarrow Xi$ continuous. The product topology is generated from the basis $\{\Pi i \in IUi : Ui \in Ti.$ such that $Xi \neq Ui$ for finitely many $i\}$. If I is a finite set then the box and product topologies are the same. The pair $(X, \tau)$ is a said to the topological product of$((Xi, \tau i))i \in I$.

**Compactness properties in topological spaces**

Let$(X, \tau)$ be a topological space and $S \subset X$. An open cover O of S is a collection of open sets that hold S, in symbols we have $S \subset \cup\{U : U \in O\}$.

Let$(X, \tau)$ be a topological space and $S \subset X$. The set S is compact in $(X, \tau)$ when all open cover has a finite subcover. That is, for any collection O that covers S, there exist U1, U2, ...Un$\in$O such that $S \subset U_{i=1}^{n} U_i$.

**Theorem**

Let(X, τ1) be a compact space, let(Y, τ2) be a topological space and let f: X → Y be a continuous function. Then f(X) is compact.

**Proof**

Let O be an open over for f(X). By using theorem (2.1.6) we know that {f−1(U) : U ∈O} is an open cover for X. Since X is compact, there is a finite subcover. There exist U1, U2, ...Un∈O such that X ⊂f−1(U1) ∪f−1(U2) ∪... ∪f−1(Un). It follows that f(X) ⊂U1 ∪U2 ∪... ∪Un. Thus, an arbitrary open cover of f(X) has a finite subcover. Hence, the theorem is proof.

**Theorem**

Let(X, τ) be a compact topological space and let Y be a closed subset of X. Then Y is a compact set.

**Proof**

Let O be an open cover for Y. Because Y is closed in X, X/Y is open in X. From this we get that O ∪X/Y is an open cover for X. Since X is compact there necessity be a finite subcover. So, there exist U1, U2, ...Un ∈U such that X ⊂U1 ∪U2 ∪... ∪Un ∪ (X/Y). It follows that Y ⊂U1 ∪U2 ∪... ∪Un. An arbitrary open cover of Y has a finite subcover, thus Y is compact.

**Theorem (Heine-Borel Theorem)**

If A subset of the Euclidean space Rn is compact if and only if it is closed and bounded.

**Definition (locally compact)**

A topological space (X, τ) is locally compact when for each x ∈ X there exists U ∈ τ with x ∈ U. So that $\overline{U}$ is compact.

**Theorem**

Let(X, τ) be a topological space. If K1, K2, ...Kn are a family of compact sets in(X, τ), then $U_{i=1}^{n}$Ki is compact.

**Proof**

Let O be an open cover for $U_{i=1}^{n}$Ki. Since Ki is a subset of $U_{i=1}^{n}$Ki for all i, then O is an open cover for every set Ki. Since each Ki is compact, there is a finite subcover of O for each i, we denote each subcover by Oi. It follows that$U_{i=1}^{n}$Qi covers$U_{i=1}^{n}$Ki. The finite union of a collection of finite sets in finite, and $U_{i=1}^{n}$Oiso we have finite subcover. Hence, the completes is proved.

**Definition (closed set)**

Let(X, τ) be a topological space. We say(X, τ) has the finite intersection property when the following holds: Let F be a family of closed sets of X with∩{F : F ∈ F} = ∅, then there exists a finite subfamily F1, F2, ...Fn of elements of F such that$\bigcap_{i=1}^{n}$∅i.

We will use a different, but equivalent, form of the finite intersection property: if, for all finite subfamilies of F we have$\bigcap_{i=1}^{n}$∅i, then ∩{F : F ∈ F}≠∅. A topological space is compact if and only if it has the finite intersection property.

**Theorem (Tychonoff′s Theorem)**

Let((Xi, τi))i∈Ibe a family of topological spaces with Xi compact for all i ∈I. Then the topological product(X, τ) is compact. Tychonoff's theorem may not hold for locally compact spaces. Instead, we have the following theorem.

**Definition (disconnected)**

A topological space(X, τ) is disconnected when there exist nonempty sets U, V ∈τ such that U ∩ V = ∅ and U ∪V = X. A set is connected when such sets don't exist. In a topological space(X, τ) a component is a connected subspace that is not properly contained in any other connected subspace of X.

**Theorem**

Given a topological space (X, τ) and a family Y of connected subspaces of X where no two elements of Y are disjoint, it follows that ∪{Y : Y ∈ Y} is connected.

**(i) Definition (homeomorphism)**

Let(X, τX) and(Y, τY) be topological spaces. A bijection f :X → Y is called a homeomorphism when equally f and f−1 are continuous. When two spaces are homeomorphic it means they share all the same topological properties; they are topologically indistinguishable. This is the topological analogy of an isomorphism of groups.

**(ii) Definition (T0 space)**

Let(X, τ) be a topological space. We call(X, τ) a T0 −space holds for all x, y ∈X with x ≠y, there exists U ∈τ such that either x ∈U and y ∉ U, or x ∈/ U and y ∈U.

**(iii) Definition (T1 space)**

Let(X, τ) be a topological space. We call(X, τ) a T1 −space when the following holds. For all x, y ∈X with x ≠y, there exist U, V ∈τ such that x ∈U, y ∉ U, y ∈V, and x ∉ V. A T2 – space is a Hausdorff space, which we have definite earlier.

**(iv) Definition**

Let G be a nonempty set and let ∗ : G × G → G be a binary operation defined by ∗(g1, g2) = (g1 ∗ g2). The pair (G, ∗) is a group if the following three properties gets:

i) For all a, b, c ∈G we have(a ∗b) ∗c = a ∗ (b ∗c) (Associativity)

ii) There exists an e ∈G such that for all a ∈G we have a∗e= e∗a= a (Identity Element)

iii) For all a ∈G there exists a−1 ∈G such that a ∗a−1 = a−1 ∗a = e (InverseElements)

(v) Definition (abelian)

Let(G, ∗) be a group. If G has the property that a ∗b = b ∗a for all a, b ∈G, then we call G abelian.

(vi) Definition (subgroup)

Let (G, ∗) be a group and H a subset of G. We call H a subgroup of G when the following holds:

i) H ≠∅,

ii) If x, y ∈H, then x ∗y ∈H (Closure under the operation of G),

iii) If x ∈H, then x−1 ∈H (Closure under inverses).

We write H ≤ G to denote subgroups.

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