International Journal on Science and Technology (IJSAT)

E-ISSN: 2229-7677 ● Website: www.ijsat.org ● Email: editor@ijsat.org

Approximation of Fixed Points of Suzuki Type Mappings in Convex G-Metric Spaces

Madhu Aggarwal

Department of Mathematics, Vaish College, Rohtak, Haryana (INDIA)
madhumdur@gmail.com

Abstract

In this paper, some results for Suzuki type mappings using Ishikawa iteration procedure are proved to approximate fixed points in convex G-metric spaces. The result generalizes various comparable results.

Keywords: fixed point, iteration procedure, Convex G-metric space.

1. Introduction and Preliminaries

In 2006, Mustafa and Sims [7] defined the concept of G metric space (as extension of the concept of metric space) as follows:

Definition 1.1.[7]. A pair $(\Omega,\tilde{\zeta})$ is said to be a **G-metric space or GM-space** if $\tilde{\zeta}:\Omega\times\Omega\times\Omega\to R^{+}$ is any function defined on a nonempty set Ω such that

(i) $\tilde{\zeta}(\varpi,\sigma,\delta)=0$ if $\varpi=\sigma=\delta$,

(ii) $\tilde{\zeta}(\varpi,\varpi,\sigma)>0\ \ \forall\varpi,\sigma\in\Omega,\varpi\neq\sigma$,

(iii) $\tilde{\zeta}(\varpi,\varpi,\sigma)\leq\tilde{\zeta}(\varpi,\sigma,\delta)\ \ \forall\varpi,\sigma,\delta\in\Omega,\sigma\neq\delta$,

(iv) $\tilde{\zeta}(\varpi,\sigma,\delta)=\tilde{\zeta}(\varpi,\delta,\sigma)=\tilde{\zeta}(\sigma,\delta,\varpi)=.....$,

(v) $\tilde{\zeta}(\varpi,\sigma,\delta)\leq\tilde{\zeta}(\varpi,\chi,\chi)+\tilde{\zeta}(\chi,\sigma,\delta),\ \ \forall\varpi,\sigma,\delta,\chi\in\Omega$,

Definition 1.2.[7] The sequence $\{g_q\}$ of points of Ω is called **G-convergent** to σ in GM- space $(\Omega,\tilde{\zeta})$ if for each $\varepsilon>0$, $\exists\, l\in N$ such that $\tilde{\zeta}(\sigma,g_q,g_r)<\varepsilon\ \ \forall q,r\geq l.$ We can say that σ is called limit of the sequence $\{\alpha_a\}$.

Definition 1.3.[7] A sequence $\{g_q\}$ in a GM-space $(\Omega,\tilde{\zeta})$ is called G-**Cauchy** if for each $\varepsilon>0$ $\exists\, l\in N$ such that $\tilde{\zeta}(g_q,g_r,g_s)<\varepsilon\ \ \forall q,r,s\geq l.$

Definition 1.4.[7] If every G-Cauchy sequence of points of a GM- space $(\Omega,\tilde{\zeta})$ is G-convergent then GM- space $(\Omega,\tilde{\zeta})$ will be called **G- complete.**

Lemma 1.5.[7] Let $(\Omega,\tilde{\zeta})$ be a GM-space, then

$$\tilde{\zeta}(\varpi,\sigma,\sigma)\leq 2\tilde{\zeta}(\sigma,\varpi,\varpi)\qquad\forall\varpi,\sigma\in\Omega.$$

Many authors have obtained fixed point results for various contractive conditions in GM- spaces [2,3,5,6,8-11]. In 2012, Aggarwal et.al.[1] obtained fixed point results for Suzuki Type contractions in GM- spaces.

Takahashi [12] defined the concept of convexity in metric space and proved several fixed-point results for nonexpansive mappings. Recently, Yildirim and Khan [13] defined convex GM- space as follows:

Definition 1.6.[13]. A function $\tilde{W}:\Omega\times\Omega\times I\times I\to\Omega$ defined on a GM-space $(\Omega,\tilde{\zeta})$ is said to be a **convex structure** on $(\Omega,\tilde{\zeta})$ if it satisfies $\tilde{\zeta}(\tilde{W}(\omega,\tau;i,j),o,\upsilon)\le i\tilde{\zeta}(\omega,o,\upsilon)+j\tilde{\zeta}(\tau,o,\upsilon)$ $\forall\omega,\tau,o,\upsilon\in P$ and $g,u\in I=[0,1]$ satisfying $g+u=1$.

Then $(\Omega,\tilde{\zeta},\tilde{W})$ is said to be a **convex GM- space.**

Definition 1.7.[13] A nonempty subset **R** is said to be convex subset of convex GM-space $(\Omega,\tilde{\zeta},\tilde{W})$ if

$\tilde{W}(\varpi,\sigma;g,u)\in R$ $\forall\varpi,\sigma\in R$ and $g,u\in I=[0,1]$.

Also, Yildirim and Khan [13] transformed the Mann iterative procedure in convex G-metric space as follows:

Definition 1.8.[13] Let $\Gamma:\Omega\to\Omega$ be a mapping on a convex GM-space $(\Omega,\tilde{\zeta},\tilde{W})$. Let $\{g_q\}$ be any two sequences on [0,1] for $q\in N$. Then for any $\varpi_0\in\Omega$ and $q\in N$,the **Mann iterative procedure** is defined by the sequence $\{\varpi_q\}$ as

$$\text{(1.1.1)}\qquad \varpi_{q+1}=\tilde{W}(\varpi_q,\Upsilon\varpi_q;1-g_q,g_q)$$

Now, we will transform the Ishikawa iterative procedure in convex G-metric space as follows:

Definition 1.9. Let $\Gamma:\Omega\to\Omega$ be a mapping on a convex GM-space $(\Omega,\tilde{\zeta},\tilde{W})$. Let $\{g_q\},\{u_q\}$ be any two sequences on [0,1] for $q\in N$. Then for any $\varpi_0\in\Omega$ and $q\in N$, the **Ishikawa iterative procedure** is defined by the sequence $\{\varpi_q\}$ as

$$\text{(1.1.2)}\qquad \varpi_{q+1}=\tilde{W}(\varpi_q,\Gamma\sigma_q;1-g_q,g_q),\ \ \sigma_q=\tilde{W}(\varpi_q,\Gamma\varpi_q;1-u_q,u_q), \qquad .$$

It should be noted here that if we choose $u_q=0$ in (1.1.2) then we get (1.1.1). Thus, Mann iterative procedure is a special case of Ishikawa iterative procedure.

In the next section, some results for approximating fixed points in convex G-metric spaces are proved for Suzuki type mapping using Ishikawa Iteration procedure. The main result generalizes the results of [1,4,9].

2. Main Result

Theorem 2.1. Let $(\Omega,\tilde{\zeta},\tilde{W})$ be a convex GM-space and $\Gamma:\Omega\to\Omega$ be a map with $F(\Gamma)\neq\phi$. Define a strictly increasing function $\kappa:[0,\frac{1}{4})\to(\frac{4}{5},1]$ *by* $\kappa(\iota)=\dfrac{1}{1+\iota}$. Then, if there exists $\iota\in[0,\frac{1}{4})$ such that $\forall\varpi,\sigma\in\Omega$

$$\text{(2.1.1)}\qquad \kappa(\iota)\tilde{\zeta}(\varpi,\Gamma\varpi,\Gamma\varpi)\le\tilde{\zeta}(\varpi,\sigma,\sigma)\ \text{ implies }\ \tilde{\zeta}(\Gamma\varpi,\Gamma\sigma,\Gamma\sigma)\le\iota\tilde{\zeta}(\varpi,\sigma,\sigma)\ .$$

Let $\varpi_0 \in \Omega$ and $\{\varpi_q\}$ be Ishikawa iterative procedure defined by (1.1.2), where $\sum_{q=0}^{\infty} g_q = \infty.$ Then, $\{\varpi_q\}$ converges strongly to a fixed point of Γ .

Proof. Let $\vartheta \in F(\Gamma),$ then

(2.1.2) $$\tilde{\zeta}(\varpi_{q+1}, \vartheta, \vartheta) = \tilde{\zeta}\left(\tilde{W}(\varpi_q, \Gamma\sigma_q; 1-g_q, g_q), \vartheta, \vartheta\right) \leq \left(1-g_q\right)\tilde{\zeta}(\varpi_q, \vartheta, \vartheta) + g_q\tilde{\zeta}(\Gamma\sigma_q, \vartheta, \vartheta).$$

Now,

(2.1.3) $$\tilde{\zeta}(\Gamma\sigma_q, \vartheta, \vartheta) = \tilde{\zeta}(\Gamma\sigma_q, \Gamma\vartheta, \Gamma\vartheta) \leq 2\tilde{\zeta}(\Gamma\vartheta, \Gamma\sigma_q, \Gamma\sigma_q).$$

Since,

(2.1.4) $\kappa(\iota)\tilde{\zeta}(\vartheta, \Gamma\vartheta, \Gamma\vartheta) \leq \tilde{\zeta}(\vartheta, \sigma_q, \sigma_q)$ implies $\tilde{\zeta}(\Gamma\vartheta, \Gamma\sigma_q, \Gamma\sigma_q) \leq \iota\tilde{\zeta}(\vartheta, \sigma_q, \sigma_q)$

Using (2.1.4) in (2.1.3), we have

(2.1.5) $$\tilde{\zeta}(\Gamma\sigma_q, \vartheta, \vartheta) \leq 2\iota\tilde{\zeta}(\vartheta, \sigma_q, \sigma_q) \leq 4\iota\tilde{\zeta}(\sigma_q, \vartheta, \vartheta).$$

Let $\wp = 4\iota.$ Then, $0 \leq \iota < \frac{1}{4}$ gives $0 \leq \wp < 1$.

So, (2.1.5) becomes

(2.1.6) $$\tilde{\zeta}(\Gamma\sigma_q, \vartheta, \vartheta) \leq \wp\tilde{\zeta}(\sigma_q, \vartheta, \vartheta).$$

Thus, combining (2.1.2) and (2.1.6), we get

(2.1.7) $$\tilde{\zeta}(\varpi_{q+1}, \vartheta, \vartheta) \leq \left(1-g_q\right)\tilde{\zeta}(\varpi_q, \vartheta, \vartheta) + g_q\wp\tilde{\zeta}(\sigma_q, \vartheta, \vartheta).$$

Now,

(2.1.8) $$\tilde{\zeta}(\sigma_q, \vartheta, \vartheta) = \tilde{\zeta}\left(\tilde{W}(\varpi_q, \Gamma\varpi_q; 1-u_q, u_q), \vartheta, \vartheta\right) \leq \left(1-u_q\right)\tilde{\zeta}(\varpi_q, \vartheta, \vartheta) + u_q\tilde{\zeta}(\Gamma\varpi_q, \vartheta, \vartheta).$$

Now,

(2.1.9) $$\tilde{\zeta}(\Gamma\varpi_q, \vartheta, \vartheta) = \tilde{\zeta}(\Gamma\varpi_q, \Gamma\vartheta, \Gamma\vartheta) \leq 2\tilde{\zeta}(\Gamma\vartheta, \Gamma\varpi_q, \Gamma\varpi_q).$$

Since,

(2.1.10) $\kappa(\iota)\tilde{\zeta}(\vartheta, \Gamma\vartheta, \Gamma\vartheta) \leq \tilde{\zeta}(\vartheta, \varpi_q, \varpi_q)$ implies $\tilde{\zeta}(\Gamma\vartheta, \Gamma\varpi_q, \Gamma\varpi_q) \leq \iota\tilde{\zeta}(\vartheta, \varpi_q, \varpi_q)$

Using (2.1.10) in (2.1.9), we have

(2.1.11) $$\tilde{\zeta}(\Gamma\varpi_q, \vartheta, \vartheta) \leq 2\iota\tilde{\zeta}(\vartheta, \varpi_q, \varpi_q) \leq 4\iota\tilde{\zeta}(\varpi_q, \vartheta, \vartheta).$$

Using $\wp = 4\iota,$ (2.1.11) becomes

(2.1.12) $$\tilde{\zeta}(\Gamma\varpi_q, \vartheta, \vartheta) \leq \wp\tilde{\zeta}(\varpi_q, \vartheta, \vartheta).$$

Thus, combining (2.1.8) and (2.1.12), we get

(2.1.13) $$\tilde{\zeta}(\sigma_q, \vartheta, \vartheta) \leq \left(1-u_q\right)\tilde{\zeta}(\varpi_q, \vartheta, \vartheta) + u_q\wp\tilde{\zeta}(\varpi_q, \vartheta, \vartheta) \quad \left[1-u_q\left(1-\wp\right)\right]\tilde{\zeta}(\varpi_q, \vartheta, \vartheta).$$

Using (2.1.7) and (2.1.13), we get

(2.1.14) $\tilde{\zeta}(\varpi_{q+1},\vartheta,\vartheta)$

$$\le\left(1-g_q\right)\tilde{\zeta}(\varpi_q,\vartheta,\vartheta)+g_q\wp\left[1-u_q(1-\wp)\right]\tilde{\zeta}(\sigma_q,\vartheta,\vartheta)$$
$$=\left[1-g_q(1-\wp)-g_qu_q\wp(1-\wp)\right]\tilde{\zeta}(\varpi_q,\vartheta,\vartheta)$$
$$\le\left[1-g_q(1-\wp)\right]\tilde{\zeta}(\varpi_q,\vartheta,\vartheta)$$
$$\le\prod_{h=0}^{q}[1-g_h(1-\wp)]\tilde{\zeta}(\varpi_0,\vartheta,\vartheta)$$
$$\le e^{-(1-\wp)\sum_{h=0}^{q}g_h}\tilde{\zeta}(\varpi_0,\vartheta,\vartheta).$$

Since, $\sum_{q=0}^{\infty}g_q=\infty$ and $0\le\wp<1$, this gives $e^{-(1-\wp)\sum_{h=0}^{q}g_h}\to 0.$

Thus, $\lim_{q\to\infty}\tilde{\zeta}(\varpi_q,\vartheta,\vartheta)=0.$

Hence, the sequence $\{\varpi_q\}$ converges strongly to a fixed point ϑ of Γ.

Corollary 2.2 [4] Let $(\Omega,\tilde{\zeta},\tilde{W})$ be a convex GM-space and $\Gamma:\Omega\to\Omega$ be a map with $F(\Gamma)\ne\phi$. Define a

strictly increasing function $\kappa:[0,\frac{1}{4})\to(\frac{4}{5},1]$ *by* $\kappa(\iota)=\dfrac{1}{1+\iota}$. Then, if there exists $\iota\in[0,\frac{1}{4})$ such that $\forall\,\varpi,\sigma\in\Omega$

(2.2.1) $\kappa(\iota)\tilde{\zeta}(\varpi,\Gamma\varpi,\Gamma\varpi)\le\tilde{\zeta}(\varpi,\sigma,\sigma)$ implies $\tilde{\zeta}(\Gamma\varpi,\Gamma\sigma,\Gamma\sigma)\le\iota\tilde{\zeta}(\varpi,\sigma,\sigma)$.

Let $\varpi_0\in\Omega$ and $\{\varpi_q\}$ be Mann iterative procedure defined by (1.1.1), where $\sum_{q=0}^{\infty}g_q=\infty$. Then, $\{\varpi_q\}$ converges strongly to a fixed point of Γ .

Proof. Choose $u_q=0$ in (1.1.2) to get (1.1.1).

Corollary 2.2 gives approximation result for the mapping used in Theorem 3.1 of [1].

Corollary 2.3 Let $(\Omega,\tilde{\zeta},\tilde{W})$ be a convex GM-space and $\Gamma:\Omega\to\Omega$ be a map with $F(\Gamma)\ne\phi$. Then, if there exists $\iota\in[0,\frac{1}{4})$ such that $\forall\,\varpi,\sigma\in\Omega$

(2.3.1) $\tilde{\zeta}(\Gamma\varpi,\Gamma\sigma,\Gamma\sigma)\le\iota\tilde{\zeta}(\varpi,\sigma,\sigma)$.

Let $\varpi_0\in\Omega$ and $\{\varpi_q\}$ be Ishikawa iterative procedure defined by (1.1.1), where $\sum_{q=0}^{\infty}g_q=\infty$. Then, $\{\varpi_q\}$ converges strongly to a fixed point of Γ .

Proof. The result follows directly from theorem 2.1.

Corollary 2.4 Let $(\Omega,\tilde{\zeta},\tilde{W})$ be a convex GM-space and $\Gamma:\Omega\to\Omega$ be a map with $F(\Gamma)\ne\phi$. Then, if there exists $\iota\in[0,\frac{1}{4})$ such that $\forall\,\varpi,\sigma\in\Omega$

(2.4.1) $\tilde{\zeta}(\Gamma\varpi,\Gamma\sigma,\Gamma\sigma)\le\iota\tilde{\zeta}(\varpi,\sigma,\sigma)$.

Let $\varpi_0 \in \Omega$ and $\{\varpi_q\}$ be Mann iterative procedure defined by (1.1.1), where $\sum_{q=0}^{\infty} g_q = \infty$. Then, $\{\varpi_q\}$ converges strongly to a fixed point of Γ .

Proof. Choose $u_q = 0$ in (1.1.2) to get (1.1.1).

Corollary 2.4 gives approximation result for the mapping used in Theorem 5.1.7 of [9].

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