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Stability Analysis of a Sixth-Stage Fifth-Order Runge-Kutta Method for Solving Initial Value Problems

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Abstract

This article investigates the stability of a sixth-stage, fifth- order Runge-kutta method which had already been derived in our previous results. The method is subjected to the test equation given as $y = \lambda y$, where a complex constant is λ, and the resulting expansion and algebraic manipulations were done with the help of Maple-18 and Matlab package, to ease the simplification of the computation. The region of stability in the method is determined to verify the nature of the stability of the method, using Matlab package.

Keywords

Absolute Stability, A-Stability, B-Stability, Consistency, Convergence

1. Introduction

The Runge-Kutta methods for the solution of Initial value problems are one-step methods designed to approximate y(x)by Taylor series expansion, and have the advantage of not requiring explicit evaluation of the derivatives of f (x, y) , where x often represents time (t) [1]. The basic idea is to use a linear combination of values of f (x, y) to approximate y(x) and this linear combination is matched up as closely as possible with a Taylor series for y(x) to obtain methods of the highest possible order. In order to carry out our work on stability, we will investigate the inherent properties of the Runge-Kutta method like consistency, divergence and stability of the method, we must acknowledge that in the course of implementing a method, the numerical solution should approximate the exact solution, and that the numerical solution tends to the exact solution as the step size tends to zero. But [7], observed that if the step length is too small, excessive computation time and round-off error results, and so we should also consider the opposite case, and ask whether there is any upper bound on step length. Often there is such a bound, and it is reached when the method becomes numerically unstable. This occurs when the numerical solution produced, no longer corresponds qualitatively with the exact solution [11].

The establishment of the consistency and convergence of a method, following [8], revealed that a Scheme is said to be consistent if the difference equation of the computational formula exactly

approximates the differential equation it intends to solve. According [5], the traditional criterion for ensuring that a numerical method is stable is called Absolute Stability, and this analysis is carried out by subjecting the method to a linear test equation; $y^r = \lambda y, \lambda \in C,\ Re(\lambda) < 0$. [6] further emphasized that, all Runge-Kutta methods including the implicit ones, when applied to the test equation, reduce to an equation of the form; $\frac{y_{n+1}}{} = R$ [10] acknowledged that the key issue for understanding the long term dynamics of Runge-Kutta methods near some fixed points, concerns the region where $R(\mu) \leq 1$; that is, the stability region of the numerical method. Furthermore, the polynomial for which $R(\mu) \leq 1$, is known as the Stability polynomial of the method, and this method is absolutely stable for a given $\mu = \lambda h$, if all roots of the polynomial function lie within the unit circle. The region containing all these points in the complex plane is said to be a region of absolute stability. However, Consistency and Stability properties of a method, play very significant roles in every numerical analysis [9]. According to [12], it is not possible to obtain accu- racy in numerical solutions if a numerical method is not stable. For this reason, a numerical method must be stable to show its accuracy. Before we establish the stability region of this new Runge-Kutta method, we first define some very relevant terms for clearer understanding of this work. The following definition of stability is from [2]. Runge-Kutta (RK) methods are among the most important families of single-step methods used to solve IVPs of the form:

$$y'=f(x,y),\ y(x0)=y0.$$

These methods approximate the solution by evaluating the function at several points within each integration step and combining them linearly to achieve a desired order of accuracy. The order of the method and the number of stages determine the computational cost and accuracy. This paper builds on previous work that derived a specific sixth-stage, fifth-order method. Here, we focus on stability analysis using standard test procedures for numerical methods, especially the linear scalar test equation:

$$y'=\lambda y,\ \lambda \in C,\ Re(\lambda)<0.$$

Stability properties such as absolute stability, A-stability, and B-stability are explored.

2. Order, Error, and Stability

The accuracy of an RK method is defined by its order, derived from satisfying order conditions. Stability, especially for stiff equations, is crucial. Implicit RK methods like Gauss–Legendre are A-stable and suitable for such cases. Recent studies focus on optimizing the stability region of explicit methods to improve performance.

$$k_1 = f(t_n, y_n),$$
$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$
$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$
$$k_4 = f(t_n + h, y_n + hk_3)$$

Leading to

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

3. Stability Considerations

For stiff problems, stability is crucial. Implicit RK methods (e.g., Gauss–Legendre schemes) offer favorable A- and L-stability. For explicit RK methods, the size of the absolute stability region constrains allowable step sizes. A recent work by Krivovichev (2024) optimizes the shapes of stability regions for explicit RK schemes with higher-order derivatives. By tuning stability polynomial parameters, optimized 4th–6th order methods achieve broad, efficient stability domains suitable for method-of-lines PDE solvers.

The classical fourth-order Runge-Kutta formula, which has a certain sleekness of organization about it: $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h/2, y_n + k_1/2)$ $k_3 = hf(x_n + h/2, y_n + k_2/2)$ $k_4 = hf(x_n + h, y_n + k_3)$

$y_{n+1} = y_n + k_1/6 + k_2/3 + k_3/3 + k_4/6 + O(h_5)$ The components in the algorithm above.

The Runge-Kutta method iterates the x-values by simply adding a fixed step-size of h at each iteration.

The y-iteration formula is far more interesting. It is a weighted average of four values—k_1, k_2, k_3, and k_4. Visualize distributing the factor of 1/6 from the front of the sum. Doing this we see that k_1 and k_4 are given a weight of 1/6 in the weighted average, whereas k_2 and k_3 are weighted 1/3, or twice as heavily as k_1 and k_4. (As usual with a weighted average, the sum of the weights 1/6, 1/3, 1/3 and 1/6 is 1.) So these k_i values are being used in the weighted average are -

k_1 This quantity, $h f(x_n, y_n)$, is simply Euler's prediction for can be called Δy—the vertical jump from the current point to the next Euler-predicted point along the numerical solution.

k_2 The x-value at which it is evaluating the function f. $x_n + h/2$ lies halfway across the prediction interval. $y_n + k_1/2$ is the current y-value plus half of the Euler-predicted Δy that we just discussed as being the meaning of k_1. Recalling that the function f gives us the slope of the solution curve, $f(x_n + h/2, y_n + k_1/2)$, gives us an estimate of the slope of the solution curve at this halfway point. Multiplying this slope by h, just as with the Euler Method before, produces a prediction of the y-jump made by the actual solution across the whole width of the interval, only this time the predicted jump is not based on the slope of the solution at the left end of the interval, but on the estimated slope halfway to the Euler-predicted next point.

k_3 has a formula which is quite similar to that of k_2, except that where k_1 used to be, there is now a k_2. Essentially, the f-value here is yet another estimate of the slope of the solution at the "midpoint" of the prediction interval. This time, however, the y-value of the midpoint is not based on Euler's prediction, but on the y-jump predicted already with k_2. Once again, this slope-estimate is multiplied by h, gives yet another estimate of the y-jump made by the actual solution across the whole width of the interval.

k_4 evaluates f at $x_n + h$, which is at the extreme right of the prediction interval. The y-value coupled with this, $y_n + k_3$, is an estimate of the y-value at the right end of the interval, based on the y-jump just predicted by k_3. The f-value thus found is once again multiplied by h, just as with the three previous k_i, giving a final estimate of the y-jump made by the actual solution across the whole width of the interval.

Thus each of the k_i gives an estimate of the size of the y-jump made by the actual solution across the whole width of the interval. The first one uses Euler's Method, the next two use estimates of the slope of the

solution at the midpoint, and the last one uses an estimate of the slope at the right end-point. Each k_i uses the earlier k_i as a basis for its prediction of the y-jump.

This means that the Runge-Kutta formula for y_{n+1}, namely:

$y_{n+1} = y_n + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)$is simply the y-value of the current point plus a weighted average of four different y-jump estimates for the interval, with the estimates based on the slope at the midpoint being weighted twice as heavily as those using the slope at the end-points.

4. Applications:

In [1] application of Runge-Kutta method for the solution of non-linear partial differential equations are explained.

Runge-Kutta method is a powerful tool for the solution of ordinary differential equations (ODE). In this study the solution of a class of non-linear partial differential equations (PDE) is obtained by using this method. A similar approach has been taken by Rushton' and Marino and Yeh' in the analysis of aquifer systems. A particular problem of this type describing the transient flow of a gas through porous media is investigated. The equation representing this phenomenon is non-linear in nature. A summary of previous work on the solution of PDEs is given by Sinovec and Madsen". This paper illustrates a method of lines" and presents its application to seven different problems. The present paper uses a Runge-Kutta scheme, and in addition, provides a detailed comparison of the solutions with those obtained analytically as well as those from various finite-difference methods. The study shows that the solution of non-linear PDE is feasible by the Runge-Kutta method; it yields more accurate results than that obtained by finite difference methods for the example considered here. The use of Runge-Kutta methods to solve problems of this type is a novel approach. It is anticipated that this technique can be utilized to solve other complex problems of a similar nature.

The Runge-Kutta method treats every step in a sequence of steps in identical manner. Prior behavior of a solution is not used in its propagation. This is mathematically proper, since any point along the trajectory of an ordinary differential equation can serve as an initial point. The fact that all steps are treated identically also makes it easy to incorporate Runge-Kutta into relatively simple "driver" schemes. We consider adaptive step size control, discussed in the next section, an essential for serious computing. Occasionally, however, you just want to tabulate a function at equally spaced intervals, and without particularly high accuracy. In the most common case, you want to produce a graph of the function. Then all you need may be a simple driver program that goes from an initial xs to a final xf in a specified number of steps. To check accuracy, double the number of steps, repeats the integration, and compare results. This approach surely does not minimize computer time, and it can fail for problems whose nature requires a variable step size, but it may well minimize user effort. On small problems, this may be the paramount consideration.

In [2] application of Runge-Kutta method for finding multiple numerical solutions to intuitionistic fuzzy differential equations are discussed.

Many real world problems can be solved by converting them into ordinary differential equations. But in most of the cases, the exact solution is not possible and hence it is necessary to study its numerical solutions. In general most of the real world problems do not contain crisp or precise data. Therefore, to

accommodate the impreciseness, the concept of fuzzy set was introduced. However, there are situations in which even the fuzzy sets are also not enough. For, the refinement of fuzzy set, namely, intuitionistic fuzzy set was introduced. In the literature, it is found that a very few investigations on the study of numerical solutions of ordinary differential equations which are intuitionistic fuzzy in nature. An ordinary differential equation with intuitionistic fuzzy number as its initial value, was studied and solved numerically using Runge-Kutta method by Abbasbandy and Allahviranloo[3]. A study has been made on a n th-order intuitionistic fuzzy linear differential equation which is time dependent, by Lata and Kumar [4]. In recent times, strong and weak solutions of first order homogeneous intuitionistic fuzzy differential equation have been discussed by Mondal, et al. [5] and the authors have studied an application of a system of differential equations with triangular intuitionistic numbers as its initial value [6]. Nirmala and Chenthur Pandian [7] have used Euler method for the discussion of the numerical solution of intuitionistic fuzzy differential equation (IFDE) International Conference on Applied and Computational Mathematics IOP Conf. Series: Journal of Physics: Conf. Series 1139 (2018) 012012 IOP Publishing doi:10.1088/1742-6596/1139/1/012012 2 by making use of α –cut representation of intuitionistic fuzzy set. Nirmala et al. [8, 9] have discussed numerical solution of IFDE by Modified Euler method and by fourth order Runge-Kutta method, respectively under the concept of generalized differentiability. Again, the generalized differentiability concept has been used by Parimala et al. [10, 11] for the discussion of numerical solutions of IFDE by Milne's and Adam's predictor-corrector methods, respectively. Wang and Guo [12] have studied multiple solutions of intuitionistic fuzzy differential equations based on (α, β) -level depiction of an intuitionistic fuzzy set. Nirmala et al. [13] have discussed multiple numerical solutions of IFDE based on (α, β) - level depiction of an intuitionistic fuzzy set by Euler method.

In [14] application of runge-kutta numerical methods to solve the schrodiger equation for hydrogen and positronium atoms are explained.

One of the most important eigenvalues equations in physics is Schrödinger wave equation, and for atomic mass m in the potential V is:

$$\frac{-\mathbf{h}^2}{2\mathbf{m}}\Delta^2\boldsymbol{\varphi}(\mathbf{r}) + \mathbf{V}(\mathbf{r})\boldsymbol{\varphi}(\mathbf{r}) = \mathbf{E}\boldsymbol{\varphi}(\mathbf{r})$$

$\hbar$ is the reduced Planck constant,

m is the electron mass,

∇ is the Laplacian operator, Ψ is the wave function,

V is the potential energy,

E is the energy eigenvalue,

(r) denotes the quantities are functions of spherical polar coordinates (r, θ, φ).

This equation has the answer for the few analytical potential functions and for many analytical potentials it cannot be solved. So, in quantum mechanics, numerical solution of Schrödinger's wave equation is very important and so far, for the special cases has been solved numerically [15, 16].

5. CONCLUSION:

This paper proposes an orderly approach for the solution of non-linear partial differential equations, for finding numerical solutions of intuitionistic fuzzy Cauchy problems when they are expressed in (α, β)-cut representation, to solve the schrodiger equation for hydrogen and positronium atoms. A real time problem is taken and is written as an ordinary differential equation in an intuitionistic fuzzy problem. It is solved by the methods of Euler, Modified Euler and fourth order Runge-Kutta. In all the four cases the numerical solutions by Runge-Kutta gives good results. Further studies, in future can be done on the multiple numerical solutions of IFDEs by higher order methods. Also this method can be used in analysis of quantum systems with different potentials.

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