International Journal on Science and Technology (IJSAT)

E-ISSN: 2229-7677 ● Website: www.ijsat.org ● Email: editor@ijsat.org

Methods of Fractional Differential Equations in Algebra and Calculus

Konthoujam Ibochouba Singh[1], Md. Indraman Khan[2], Prof. Irom Tomba Singh[3]

[1,2,3]Department of Mathematics
[1,3]Manipur International University, Imphal
[2]Pettigrew College, Ukhrul, Manipur
[1]marcoathokpam@gmail.com

Abstract:

This paper introduces a novel approach, the Inverse Fractional Shehu Transform Method, for solving both homogeneous and non-homogeneous linear fractional differential equations. The fractional derivatives are considered in the Riemann-Liouville and Caputo senses. By applying the Laplace transform and the convolution product to the Riemann-Liouville fractional of matrices, we obtain accurate solutions for systems of matrix fractional differential equations. The method's effectiveness is demonstrated through examples, and its accuracy is verified by comparing the results with existing solutions in the literature. A numerical algorithm of fractional differential algebraic equations in terms of the theory of sliding mode control and the Grünwald-Letnikov is proposed assuming sliding mode surface.

Keywords: Inverse Fractional Shehu transform, Riemann-Liouville fractional derivative, Differential-Algebraic Equations, Sliding Mode Control, Laplace Transform.

1. Introduction:

Fractional differential equations (FDEs) have gained significant attention in applied mathematics and physics due to their ability to model complex phenomena in various fields such as electrical circuits, fluid mechanics, diffusion processes, relaxation phenomena, damping laws, and mathematical biology. To solve these equations, several mathematical techniques have been developed, including the Variational Iteration Method(VIM), Variational Iteration Method (VIM) [35], Adomian Decomposition Method (ADM)[8], New Iterative Method [16], Differential Transform Method , Homotopy Analysis Method (HAM) [1], Homotopy Permutation Method(HPM) [13], Fractional Reduced Differential Transform Method [18], Fractional Residual Power Series Method [19].Classical methods like the Laplace Transform Method, Fractional Green's Function Method, Mellim Transform Method and Method of Orthogonal Polynomials have also been employed [30].

In this context, the Inverse Fractional Shehu Transform Method has been proposed as an effective tool for solving linear FDEs [15]. This method utilizes operational matrices of fractional derivatives in conjunction with the Laplace transform to obtain solutions for fractional-order, multi-term differential equations. The application of this method is further extended to fractional differential-algebraic equations

(FDAEs), which are prevalent in the modeling of multi-body dynamics in complex mechanical systems [25, 23, 7]. Solving FDEs is challenging, and numerical techniques, such as those based on sliding mode control theory, have been developed to address this issue. In engineering applications, systems described by FDAEs often arise, particularly in the context of fractional chaotic systems controlled by sliding mode models. These systems are characterized by FDAEs, with the sliding surface corresponding to algebraic equations. Applications of FDAEs are found in fields such as electrochemistry, fractional control systems, and biochemistry. The numerical algorithms for solving FDAEs, especially those incorporating fractional calculus, play a crucial role in addressing dynamic problems in mechanical systems and related engineering fields. In this paper, the application of the Inverse Fractional Shehu Transform Method to solve linear and nonlinear fractional differential equations is explored with a particular focus on systems involving FDAEs. We also discuss the integration of sliding mode control theory in the numerical solution of FDAEs, highlighting its effectiveness in handling uncertainties and disturbances in fractional-order systems.

Whenever a fractional chaotic system is regulated by the sliding mode model in a control theory, the control equations of motion are FDAEs and the sliding surface corresponds to the algebraic equations. FDAEs can be involved /employed in fields such as electrochemistry, fractional controller, biochemistry etc [10]. Several algebraic systems mostly approaching from engineering applications and is mostly applied for describing dynamic problems. Using FDAEs recently with the application of fractional order theory in engineering is becoming an interest task for describing dynamic problems in mechanical systems and engineering related fields. Like DAEs in aerospace, robot and engineering fields, the numerical algorithm of FDAEs working in association of fractional calculus and engineering technology will take an important part. For the solution of FDAEs, certain nonlinear control technologies are also employed successfully. Wang et al. [34] on the basis of sliding mode control theory presented a numerical method of FDAEs.

By applying various methods such as variational iteration method [26, 29], Spectral methods [24], Adomian Decomposition method [27], Homotopy perturbation method [11] etc the fractional calculus operation have been analyzed as PDEs, fractional integro-differential equations and dynamic systems. For fractional calculus of the matrices type several operations are analyzed convolution product is developed to the Riemann-Liouville fractional integral of matrices [21]. The Legendre operational matrix to the fractional differential equations for linear and non-linear are universal. Further combination of Legendre series with the Legendre operational matrix of fractional derivative are employed in Caputo sense for the numerical integration of FDEs [32]. With the presentation of new shifted Chebyshev operational matrix of fractional integration in the Riemann-Liouville sense for evaluating linear, multi-term fractional differential equations is forwarded by applying it with spectral Tau method [3]. On the basis of operational matrices of differintegrals for various kinds of fractional differential equation for linear and non-linear spectral tools are analysed [4]. Approximation and numerical methods are applied as most fractional differential equations do not give exact analytic solutions. For the fractional differential equations the numerical solutions related to finite difference methods and several spectral algorithm for FDE were declared [5]. Bhrawy [5] proposed an operational matrix formulation of the collecting technique for 1 and 2-dimensional non-linear fractional sub-diffusion equations.

Definitions and Preliminaries:

Various definitions of a fractional derivative of order $\alpha \geq 0$ are used amongst which commonly used are the Riemann-Liouville and Caputo.

Definition 1.1

For a function f $\in C_{\mu}, \mu \geq -1$, the Riemann- Liouville fractional integral operator I^{α} of order α is defined as the following:

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \alpha > 0, t > 0, \\ f(t), \qquad \alpha = 0, \end{cases}$$

where $\Gamma(.)$ Is the Gamma function.

Definition 1.2

The Riemann- Liouville fractional derivative operator ${}^{R}D^{\alpha}$ of order α for a function f $\in C_{\mu}, \mu \geq -1$, is given as

$${}^{R}D^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_0^t (t-\xi)^{n-\alpha-1} f(\xi) d\xi \ , t > 0,$$

Where $n-1 < \alpha \leq n, n \in N$.

Definition 1.3

The fractional derivative of f(t) in the Caputo sense is defined as the following

$${}^{c}D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \ , t > 0,$$

Where $n-1 < \alpha \leq n, n \in N$, f$\in$ C_{-1}^{n} [20]

Shehu transform:

Shehu transform is a new integral transform introduced by Shehu Maitama Shehu et al. (2019) for applying in solving an ordinary and partial differential equations.

Definition 1.4

For the set of functions given below

$$A = \left\{ \frac{f(t)}{\exists N}, \eta_1, \eta_2 > 0, |f(t)| < \mathrm{N}\exp\left(\frac{|t|}{\eta_j}\right), if\ t \in (-1)^{i} \times [0, \infty) \right\},$$

the Shehu transform of the function f(t) of exponential order is defined by the integral

$$\mathbb{S}[f(t)] = F(s,u) = \int_0^{\infty} e^{\left(-\frac{st}{u}\right)} f(t) dt, t > 0,$$

Inverse Shehu transform:

For finding the inverse Shehu transform function f (t)= $\mathbb{S}^{-1}[F(s,u)]$, the following theorems are important.

Theorem 1: If $\alpha,\beta > 0, a \in \mathbb{R}$, and $|a| < \frac{s^\alpha}{u^\alpha}$, then the inverse Shehu transform formula is

$$\mathbb{S}^{-1}\left[\frac{u^\beta s^{\alpha-\beta}}{s^\alpha + a u^\alpha}\right] = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)$$

Theorem 2:

If $\alpha \geq \beta > 0, a \in \mathbb{R}$, and $|a| < \left(\frac{s}{u}\right)^{\alpha-\beta}$, then the inverse Shehu transform formula is

$$\mathbb{S}^{-1}\left[\frac{u^{(n+1)(\alpha+\beta)}}{\left(s^\alpha u^\beta + a u^\alpha s^\beta\right)^{n+1}}\right] = t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}$$

Theorem 3:

If $\alpha \geq \beta, \alpha > \gamma, a \in \mathbb{R}, |a| < \left(\frac{s}{u}\right)^{\alpha-\beta}, and |b| < \frac{s^\alpha u^\beta + a u^\alpha s^\beta}{u^{\alpha+\beta}}$, then the inverse Shehu transform formula is

$$\left[\frac{u^{\alpha+\beta-\gamma}s^\gamma}{s^\alpha u^\beta + a u^\alpha s^\beta + b u^{\alpha+\beta}}\right] = t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}$$

Property (1):

The Shehu transform is a linear operator.

Property (2):

Let us suppose that F(s, u) and G (s, u)be the Shehu transforms of f(t) and g(t), both defined on a set $A = \left\{\frac{f(t)}{\exists N}, \eta_1, \eta_2 > 0, |f(t)| < \mathrm{N}\exp\left(\frac{|t|}{\eta_j}\right), if\ t \in (-1)^i \times [0,\infty)\right\}$.

Then the Shehu transform of their convolution is defined as

$$\mathbb{S}[(f * g)(t)] = F(s,u)G(s,u),$$

where the convolution of the two functions is given by

$$(f * g)(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi$$

Property (3):

The Shehu transform of t^α is given by

$$\mathbb{S}[t^\alpha] = \left(\frac{u}{s}\right)^{\alpha+1}\Gamma(\alpha+1)$$

Shehu transform for fractional derivatives:

For the function f(t) of order α, if F(s, u) be the Shehu transform of f(t), then the Shehu transform of the Riemann-Liouville fractional integral is given as

$$\mathbb{S}[I^{\alpha}f(t)] = \left(\frac{u}{s}\right)^{\alpha} F(s,u)$$

Proof:

For a function f $\in C_{\mu}, \mu \geq -1$, the Riemann- Liouville fractional integral operator I^{α} of order α is defined as the following:

$$I^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1} f(\xi)d\xi, \alpha > 0, t > 0, \\ f(t), \qquad \alpha = 0, \end{cases} \quad (1)$$

Where $\Gamma(.)$ Is the Gamma function.

As in equation (1), the Riemann-Liouville fractional integral for the function f(t) can be given as convolution

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1\,\zeta} * f(t)$$

Using property (2) and (3) and applying the Shehu transform above, the following result is obtained as

$$\mathbb{S}[I^{\alpha}f(t)] = \mathbb{S}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)\right] = \mathbb{S}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{S}[f(t)] = \left(\frac{u}{s}\right)^{\alpha} F(s,u)$$

The application of the inverse fractional Shehu transform method to some linear fractional differential equations [20].

Example 1.

For the following linear fractional initial value problem [22]

$${}^{R}D^{\frac{1}{2}}y(t) + y(t) = 0, \quad (2)$$

Subject to the initial condition

$$\left[{}^{R}D^{-1/2}y(t)\right]_{t=0} = 2. \quad (3)$$

By a theorem and applying the Shehu Transform on both sides of (2), we have

$$\left(\frac{s}{u}\right)^{\frac{1}{2}} Y(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{k}\left[{}^{R}D^{\frac{1}{2}-k-1}f(t)\right]_{t=0} + Y(s,u) = 0. \quad (4)$$

Using (3) into (4), we get

$$\left[\left(\frac{s}{u}\right)^{1/2} + 1\right] Y(s,u) - 2 = 0.$$

So

$$\mathrm{Y(s, u)} = S[y(t)] = \frac{2u^{1/2}}{s^{1/2} + u^{1/2}}.$$

From the Theorem (1), the exact solution of this problem can be obtained as

y (t) $= 2t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}\left(-t^{1/2}\right).$

Example 2: For the initial value problem of non-homogeneous Bagley-Torvik equation [2]

$$y''(t) + D^{\frac{3}{2}}y(t) + y(t) = 1 + t, \tag{5}$$

subject to the initial conditions

$$y\ (0)\ =\ y'(0) = 1 \tag{6}$$

Applying the Shehu Transform on both sides of (5)and using a Theorem , we have

$$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - 1 + \frac{s^{3/2}}{u^{3/2}}Y(s,u) - \frac{s^{1/2}}{u^{1/2}}y(0) - \frac{s^{-1/2}}{u^{-1/2}}y'(0) + Y(s,u) = \frac{u}{s} + \frac{u^2}{s^2} \tag{7}$$

Substituting (6) into (7), we get

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right] = \frac{u}{s} + \frac{u^2}{s^2} + \frac{s}{u} + 1 + \frac{s^{1/2}}{u^{1/2}} + \frac{s^{-1/2}}{u^{-1/2}} \tag{8}$$

Then (8) gives

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} +\right] = \left(\frac{u}{s} + \frac{u^2}{s^2}\right)\left(\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right) \tag{9}$$

So $$Y(s,u) = S[y(t)] = \frac{u}{s} + \frac{u^2}{s^2}, \tag{10}$$

Taking the inverse Shehu transform of (10), we get

$$y\ (t) = 1 + t$$

which is the exact solution of the problem.

Example 3:

For the following linear fractional initial value problem [28]

$$^{c}D^{\alpha}y\ (t) = y(t) + 1,\ \ 0 < \alpha \le 1, \tag{11}$$

Subject to the initial condition

$$y\ (0) = 0. \tag{12}$$

Applying the Shehu transform to both sides of (11) and using Theorem , we get

$$\frac{s^\alpha}{u^\alpha}Y(s,u) = Y(s,u) + \frac{u}{s},$$

So $$Y(s,u) = S[y(t)] = \frac{u^{\alpha+1}s^{-1}}{s^\alpha + u^\alpha}$$

Applying the Theorem 1, the exact solution of this problem can be obtained as

$$y\ (t) = \ t^\alpha E_{\alpha,\alpha+1}(t^\alpha) \tag{13}$$

Fractional Differential-Algebraic Equations:

For the following affine nonlinear fractional differential-algebraic equation

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 x^{(\alpha)}(t) + f(x,t) + B_2 w(t) \\ s(x) = 0 \end{cases} \quad (14)$$

Where x $= (x_1, x_2, \dots, x_n)^T$ is the state vector, $x^{(\alpha)}(t)$ is the fractional derivatives of order α, f(x, t) is the nonlinear part, w(t) is the external input, s(x) is the algebraic equation and A, B_1, B_2 are the coefficient matrices.

Example 4:

An example with Exact Solutions [33]

Let us consider a nonlinear fractional order differential-algebraic equation

$$\begin{cases} \dot{x}_1 = -x_1 + \frac{3}{2}x_2 - x_3 + \frac{1}{\Gamma(2.5)}x_1^{0.5} + x_3^{1.5} \\ \dot{x}_2 = x_2 + \frac{1}{2}\frac{1}{x_2} - x_3^{1.5} \\ \dot{x}_3 = \frac{1}{2}x_1 + 1 - \frac{1}{2}x_3^{1.5} \\ x_1 - x_2 x_3 = 0 \end{cases} \quad (15)$$

Its exact solution are

$$\begin{cases} x_1 = t^{1.5} \\ x_2 = t^{0.5} \\ x_3 = t \end{cases} \quad (16)$$

By (14) and (15), this can be expressed in the matrix form, then we get

$$A = \begin{bmatrix} -1 & \frac{3}{2} & -1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{1}{\Gamma(2.5)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{f (x, t)} = \begin{bmatrix} x_3^{1.5} \\ \frac{1}{2}\frac{1}{x_2} - x_3^{0.5} \\ 1 - \frac{1}{2}x_3^{1.5} \end{bmatrix} \quad (17)$$

Using the sliding mode control theory, we convert the fractional order differential-algebraic equation into an equivalent fractional differential equation, and obtain the sliding mode surface, given by

$$\dot{x}(\text{t}) = Ax(t) + B_1 x^{(0.5)}(t) + f(x,t) + g(x)u$$

$$\text{s (x)} = x_1 - x_2 x_3 = 0 \quad (18)$$

Considering the controllability condition, the control matrix is chosen as g(x) $= (0,1,0)^T$.The parameters of sliding mode are set to be $\varepsilon = 5 \; and \; \lambda = 5$. The control vector u is consistent with the control law.

We start with the initial condition x $(\tau) \; = (0.001, 0.1, 0.01)^T, (\tau \leq 0)$. ( ) is solved numerically by using the present method, and the numerical solutions are studied by comparing with the exact solutions in the followings.

The Laplace transform is used for designing various engineering systems as control system in many applications

Consider fractional-order, multi-term differential equation with constant coefficients

$$\sum_{i=0}^{n} a_i D^v y(t) = D^n\left[D^{-(n-v)} y(x)\right] = D^n[D^{-u} y(x)]$$

By taking Laplace transform on the above equation, we have

$$\sum_{i=0}^{n} a_i\, L\{D^v y(x)\} = \sum_{i=0}^{n} a_i L\left\{D^n\left[D^{-(n-v)} y(x)\right]\right\} n$$

$$= \sum_{i=0}^{n} a_i s^v \mathrm{Y}(\mathrm{s}) - \sum_{i=0}^{n} a_i \sum_{i=1}^{n} s^{n-i} D^{i-1+v} y(0)$$

Using more simplifying notations, the last expression can be reduced to

$$\sum_{i=0}^{n} a_i\, L\{D^v y(x)\} = A_n(s) Y(s) - A_n(0, T)$$

We have borrowed the notation style from the soft-ware engineering concepts

$$A_n(s) = \sum_{k=0}^{n} A_n s^v$$

Therefore

$$\mathrm{Y(s)} = \frac{U(s)}{A_n(s)} - \frac{A_n(0,T)}{A_n(s)}$$

System of fractional differential equations and real symmetric matrices: [15]

For solving systems of fractional differential equations, the applications of this method with real symmetric matrices are discussed.

Let A be a real symmetric matrix .Let the following system of fractional order be considered.

$$X^\alpha = \begin{pmatrix} x_1^\alpha \\ \vdots \\ x_n^\alpha \end{pmatrix} = Ax + g(t) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

In this g is a continuous, vector-valued function of variable x for the unknown vector function $\mathrm{X} = X(t)$.

A is a diagonizable matrix as it is a symmetric matrix. Let $v_1, \ldots, v_n$ be the corresponding eigen vectors to the eigen-values $\lambda_1, \ldots, \lambda_n$.

Putting $\mathrm{x} = Py$ into the above equation gives

$$D^\alpha P y = A(Py) + g(t)$$

$$\mathrm{P} D^\alpha y = A(Py) + g(t)$$

Since P contains real numbers, then

$$D^\alpha y = P^{-1} A P y + P^{-1} g(t)$$

$$= P^T A P y + P^T g(t)$$

$$= Dy + P^T g(t)$$

$$= Dy + hg(t)$$

where D is the diagonalized matrix of A and h $= P^T g(t)$

$$\begin{pmatrix} y_1^\alpha \\ \vdots \\ y_2^\alpha \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} h_1(t) \\ \vdots \\ h_n(t) \end{pmatrix}$$

Thus

$$D^\alpha y_i = \lambda_i y_i + h_i(t)$$

By taking Laplace transform, it gives

$$\mathrm{L}\{D^\alpha y_i\} = L\{D^n[D^{-(n-\alpha)} y_i]\}$$

$$= s^n L\{D^{-(n-\alpha)} y_i\} - \sum_{i=1}^n s^{n-i} D^{i-1-(n-\alpha)}\, y_i(0)$$

$$= s^n\{s^{-(n-\alpha)} Y_i(s)\} - \sum_{i=1}^n s^{n-i} D^{i-1-(n-\alpha)}\, y_i(0)$$

$$= s^\alpha Y_i(s) - \sum_{i=1}^n s^{n-i} D^{i-1-n+\alpha}\, y_i(0)$$

By taking the inverse Laplace transform, the solution is obtained.

Consider the first order of the fractional differential equation

$$D^{0.5} y(t) + Ay(t) = u(t)$$

By taking Laplace transform on the above equation,

$$\mathrm{Y(s)} = \frac{1 - e^{-sT} - s^{0.5} e^{-sT} u(T)}{s^{0.5}(s^{0.5} + c)}$$

Since Y(s) must be analytic from the above equation at s $= -c$

$$1 - e^{-sT} + ce^{cT} u(T) = 0$$

This gives the boundary value for u (T) as

$$\mathrm{u\ (T)} = \frac{1 - e^{-cT}}{c}$$

Similarly writing the system function

$$s^{0.5} Y(s) + e^{-sT} y(T) + Ay(t) = U(s)$$

Solving for Y(s)

$$\mathrm{Y(s)} = \frac{1 - e^{-sT} - s^{0.5} e^{-sT} u(T) - s^{0.5}(s^{0.5} + c) e^{-sT} y(T)}{s^{0.5}(s^{0.5} + c)(s^{0.5} + p)}$$

Y(s) must be analytic from the above equation at s $= -p$

$$1 - e^{pT} + p^{0.5} e^{pT} u(T) + p^{0.5}(c - p) e^{pT} y(T) = 0$$

Hence

$$y\,(T) = \frac{1-e^{-pT}-p^{0.5}u(T)}{p^{0.5}(c-p)}$$

Example 5: [15] Solve the fractional-order system when $\alpha = \frac{2}{3}$

$$X^{\alpha} = \begin{pmatrix} x_1^{\alpha} \\ x_2^{\alpha} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2e^{-t} \\ 3 \end{pmatrix} = Ax + g(t)$$

Substituting x= Py forms a simple system, with P= $(v_1 \quad v_2)$

1. f values λ of A occurs when $|A-\lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda-2)^2 - 1 = (\lambda+1)(\lambda+3)$$

Hence, $\lambda = -1, -3$

In case, $\lambda = -1$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

And $v_1 = v_2$. Thus $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

In case, $\lambda = -3$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

And $v_1 = -v_2$. Thus $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Substituting $x = Py$ into the differential equation yields

$$y' = P^{-1}APy + P^{-1}g = Dy + P^{T}g$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2e^{-t} \\ 3 \end{pmatrix} = \begin{pmatrix} -y_1 + \frac{1}{\sqrt{2}}(2e^{-t}+3) \\ -3y_2 + \frac{1}{\sqrt{2}}(2e^{-t}-3) \end{pmatrix}$$

Now, first row yields

$$D^{\frac{2}{3}}y_1 = \frac{1}{\sqrt{2}}(2e^{-t}+3)$$

$$s^{\frac{2}{3}}y_1(s) - D^{-\left(1-\frac{2}{3}\right)}y_1(0) = \frac{1}{\sqrt{2}}\left(\frac{2}{s+1}+3\right)$$

$$s^{\frac{2}{3}}y_1(s) - c = \frac{1}{\sqrt{2}}\left(\frac{2}{s+1}+3\right)$$

$$y_1(x) = \frac{1}{\sqrt{2}}\left(t^{\frac{2}{3}}E_{1,\frac{5}{3}}(t) + \frac{3t^{\frac{2}{3}-1}}{\Gamma\left(\frac{2}{3}\right)}\right) + \frac{ct^{\frac{2}{3}-1}}{\Gamma\left(\frac{2}{3}\right)}$$

Similarly $y_2(x) = \frac{1}{\sqrt{2}}\left(t^{\frac{2}{3}}E_{1,\frac{5}{3}}(t) + \frac{3t^{\frac{-1}{3}}}{\Gamma\left(\frac{2}{3}\right)}\right) - \frac{ct^{\frac{-1}{3}}}{\Gamma\left(\frac{2}{3}\right)}$

2. Conclusion:

The Laplace transform method has proven to be a satisfactorily implementing and effective as well as reliable approach for solving and finding matrix fractional partial differential equations and matrix fractional differential equations. It demonstrates a high rate of convergence and practical applicability. Additionally, the inverse fractional transform method has been successfully applied to both homogeneous and non-homogeneous problems showing promise as a powerful and efficient tool for accurately solving linear fractional differential equations. Furthermore, the precision of numerical techniques for systems involving fractional differential-algebraic equations can be enhanced by utilizing the direct violation correction method, particularly when applied in conjunction with sliding mode control.

References

1. Anber, A. and Dahmani, Z. (2014). The homotopy analysis method for solving some fractional differential equations, Journal of Interdisciplinary Mathematics. Vol. 17. No. 3, pp. 255-269.
2. Bansal, M.K. and Jain, R. (2016). Analytical solution of Bagley-Torvik equation by generalize differential transform, International Journal of Pure and Applied Mathematics, Vol. 110, No. 2, pp. 265-273.
3. Bhrawy, A.H. , & Alofi, A.S. (2013). The operational matrix of fractional integration for shifted Chebyshev polynomials. Applied Mathematics Letters, 26(1), 25-31. Doi: 10.1016j.aml.2012.01.027
4. Bhrawy, A.H., Taha, T.M. & Machado, J.A.T. (2015). A review of operational matrices and spectral techniques for fractional calculus. Nonlinear Dynam,Vol. 81, pp. 1023-1052.
5. Bhrawy, A.H., Zaky, M. A, & Gorder, R. A.V. (2016). A space-time Legendre spectral tau method for the two-sided space-time Caputo fractional differential diffusion- wave equation. Numer Algorithms. Vol. 71, pp-151-180.
6. Bhrawy, A.H., Jacobi, A. (2016). Spectral collocation method for solving multi-dimensional nonlinear fractional sub-diffusion equations. Numer Algorithms, Vol. 73. Pp. 91-113.
7. Carpinelli, M.,Gubitosa, M., Mundo, D., Desmet, W. (2016).Automated independent coordinates switching for the solution of stiff DAEs with the linearly implicit Euler method.
8. Cheng, J-F. and Chu, Y-M. (2011). Solution to the linear fractional differential equation using Adomian decomposition method, Math. Probl. Eng, Vol. 2011, Article ID 587068, pp. 1-14.
9. Das S. (2011). Control system design using finite Laplace transform theory.https://arxiv.org/ftp/arxiv/papers/1101/1101.4347.pdf
10. Ghanbari, F., Ghanbari, K., Mokhtary, P. (2018).Generalized Jacobi-Galerkin method for nonlinear fractional differential algebraic equations. Comput. Appl. Math. 37, pp. 5456-5475.
11. Hashim, I.Abdulaziz, O and Momani S. (2009). Homotopy analysis method for fractional IVPs Commun Nonlinear Sci. pp. 674-684.
12. Herrmnn, R. (2024). Fractional Calculus: An Introduction for Physicists, World Scientific, New vJersey.
13. Hemed, A.A. (2014). Modified homotopy perturbation method for solving fractional differential equations, J. Appl. Math., Vol. 2014, Article ID 594245, pp. 1-9.

14. Khader, M.M. and Saad, K.M. (2018). A numerical study using Chebyshev collocation method for a problem of biological invasion: Fractional Fisher equation, Int. J. Biomath.,Vol. 11, 1850099
15. Khalouta, A. and Kadem , A. (2019). A new numerical technique for solving Caputo time- fractional biological population equation, AIMS Mathematics, Vol. 4, No. 5, pp. 1307-1319.
16. Khaloutaa, A. and Kadem, A. (2009c). Comparison of new iterative method and natural homotopy perturbation method for solving nonlinear time-fractional wave-like equations with variable coefficients, Nonlinear Dyn. Syst. Theory, Vol. 9 (1-SI), pp. 160-169.
17. Kilicman, A. , & Ahmood, W. A. (2017). On matrix fractional differential equations: Advances in Mechanical Engineering, 9(1), 168781401668335. Doi:10.1177/1687814016683359.
18. Khalouta, A. and Kadem, A. (2019b). A new representation of exact solutions for nonlinear time-fractional wave-like equations with variable coefficients. Nonlinear Dyn. Syst. Theory, Vol. 19, No. 2, pp. 319-330.
19. Khalouta, A. and Kadem , A.(2020). A new computational for approximate analytical solutions of noinlinear time-fractional wave-like equations with variable coefficients, AIMS Mathemtics, Vol. 5, No. 1, pp. 1-14.
20. Khalouta, A. Khadem, A. (2019). A New Method to solve Fractional Differential Equations: Inverse Fractional Shehu Transform Method, AAM, Vol. 14, Issue 2, pp. 926-941.
21. Kilicman, A., & AI-Zhour, Z. (2007). Kronecker operational matrices for fractional calculus and some applications. Appl Math Comp,pp. 250-265.
22. Li, Z.B. (2010). An extended fractional complex transform, Int. J. Numer. Simul., Vol. 11, pp. 335-337.
23. Louca, L.S. (2014). Modal analysis reduction of multi-body systems with generic damping. J. Comput. Sci. 5, pp. 415-426. [CrossRef]
24. Mathai, A.M., & Haubold, H.J. (2008).Special functions for applied scientists[PDF]. Doi:.1007/978-0-387-75894-7
25. Mehrmann, V., Wunderlich, L.(2009). Hybrid systems of differential- algebraic equations-Analysis and numerical solution. J.Process, Control , 19, 1218-1228.[CrossRef]
26. Momani, S., & Odibat, .(2006). Analytical approach to linear fractional partial differential equations arising in fluid mechanics. Physics Letters. A, 355 (4-5), 271-279. Doi: 10.1016/j.physleta.2006.02.048.
27. Momani, S., & Shawagfeh, N. (2006). Decomposition method for solving fractional Riccati differential equations. Applied Mathematics and Computation, 182(2), 1083-1092 doi:10.1016j. amc. 2006.05.008
28. Odibat, Z. and Momani, S. (2008). Analytical comparison between the homotopy permutation method and variational iteration method for differential equations of fractional order, Internat. J. Modern Phys, B, Vol. 22, pp. 4041-4058
29. Odibat, Z.M., & Momani, S. (2006). Application of variational iteration method to nonlinear differential equations of fractional order. International Journal of Nonlinear Sciences and Numerical Simulation, 7(1). Doi: 10.1515/ijnsns.2006.7.1.27
30. Podlubny, I. (1999). Fractional Differential Equations, Academic Press: San Diego, CA, USA, pp. 198.
31. Podlubny, I. (1999). Fractional Differential Equations , Academic Press, New York.

32. Saad, K.M., Deniz, S. and Baleanu, D. (2019). On a new modified fractional analysis of Ngumo equation, Int, J. Biomath., Vol 2, 1950034.
33. Tai, Y., Chen, N.,Wang, L., Feng, Z., & Xu, J. (2020). A numerical method for a system of fractional differential-algebraic equations based on sliding mode control. Mathematics, 8(7), 1134. Doi: 10.3390/math 8071134
34. Wang , L. Chen, N. (2014). The predictor-corrector solution for fractional order differential algebraic equation. In Proceedings of the 2014 International Conference on Fractional Differentiation and Its Applications (ICFDA), Catania, Italia, pp. 1-6.
35. Ziane, D.J. and Hamdi Cherif, M. (2018). Variational iteration transform method for fractional differential equations, Journal of Interdisciplinary Mathematics, Vol. 21, No. 1, pp. 185-199.