International Journal on Science and Technology (IJSAT)

E-ISSN: 2229-7677 ● Website: www.ijsat.org ● Email: editor@ijsat.org

Computation of Geometric-Arithmetic Uphill and Modified First Uphill Indices of Graphs

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Abstract

In this paper, we introduce the geometric-arithmetic uphill and the modified first uphill indices of a graph. Furthermore, we compute these newly defined uphill indices for some standard graphs, wheel graphs, gear graphs, helm graphs.

Keywords: geometric-arithmetic uphill index, modified first uphill index, graph.

1. Introduction

The simple graphs which are finite, undirected, connected graphs without loops and multiple edges are considered. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

A u-v path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a uphill path if for every i, $1 \le i \le k$, $d_G(v_i) \le d_G(v_{i+1})$.

A vertex v is uphill dominates a vertex u if there exists an uphill path originated from u to v. The uphill neighborhood of a vertex v is denoted by $N_{up}(v)$ and defined as: $N_{up}(v) = \{u\text{: } v \text{ uphill dominates } u\}$. The uphill degree $d_{up}(v)$ of a vertex v is the number of uphill neighbors of v, see [1,2].

In [3], Vukičević et al. introduced the geometric-arithmetic index and this index is defined as

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_G(u)d_G(v)}}{d_G(u) + d_G(v)}.$$

Motivated by the geometric-arithmetic index, the geometric-arithmetic uphill index of a graph G is defined as

$$GAU(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_{up}(u)d_{up}(v)}}{d_{up}(u) + d_{up}(v)}.$$

The first uphill index was introduced in [3] and it is defined as

$$UM_1(G) = \sum_{uv \in E(G)} \left(d_{up}(u) + d_{up}(v)\right).$$

We define the modified first uphill index of a graph G as

$$ {}^{m}UM_1(G) = \sum_{uv \in E(G)} \frac{1}{d_{up}(u) + d_{up}(v)}. $$

Recently, some uphill indices were studied such as the Nirmala uphill index [4], F-hill index [5], Sombor uphill [6], inverse sum indeg uphill index [7].

In this paper, the geometric-arithmetic uphill and the modified first uphill indices for some graphs are determined.

2. Results for Some Standard Graphs

Proposition 1. Let G be r-regular with n vertices and $r \geq 2$. Then

$$ GAU(G) = \frac{nr}{2}. $$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n-1$ for every v in G.
From definition,

$$ GAU(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_{up}(u) d_{up}(v)}}{d_{up}(u) + d_{up}(v)} = \frac{nr}{2} \frac{2\sqrt{(n-1)(n-1)}}{(n-1)+(n-1)} = \frac{nr}{2}. $$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$ GAU(C_n) = n. $$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$ GAU(K_n) = \frac{n(n-1)}{2}. $$

Proposition 2. Let G be r-regular with n vertices and $r \geq 2$. Then

$$ {}^{m}UM_1(G) = \frac{nr}{4(n-1)}. $$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n-1$ for every v in G.

$$ {}^{m}UM_1(G) = \sum_{uv \in E(G)} \frac{1}{d_{up}(u) + d_{up}(v)} $$

$$ = \frac{nr}{2} \frac{1}{(n-1)+(n-1)} $$

$$ = \frac{nr}{4(n-1)}. $$

Corollary 2.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$ {}^{m}UM_1(C_n) = \frac{n}{2(n-1)}. $$

Corollary 2.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$ {}^{m}UM_1(K_n) = \frac{n}{4}. $$

Proposition 3. Let P_n be a path with n□3 vertices. Then

$$ GAU(P_n) = \frac{4\sqrt{(n-2)(n-3)}}{2n-5} + (n-3). $$

Proof: Let P_n be a path with n□3 vertices. Clearly, P_n has two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{uv$ □ $E(P_n) \mid d_{up}(u)=n-2,\ d_{up}(v)=n-3\}$, $|E_1| = 2$.
$E_2 = \{uv$ □ $E(P_n) \mid d_{up}(u)= d_{up}(v)=n-3\}$, $|E_2| = n-3$.

$$ GAU(P_n) = \sum_{uv\in E(P_n)} \frac{2\sqrt{d_{up}(u)d_{up}(v)}}{d_{up}(u)+d_{up}(v)} $$
$$ = 2\frac{2\sqrt{(n-2)(n-3)}}{(n-2)+(n-3)} + (n-3)\frac{2\sqrt{(n-3)(n-3)}}{(n-3)+(n-3)} $$
$$ = \frac{4\sqrt{(n-2)(n-3)}}{2n-5} + (n-3). $$

Proposition 4. Let P_n be a path with n□3 vertices. Then

$$ {}^{m}UM_1(P_n) = \frac{2}{2n-5} + \frac{1}{2}. $$

Proof: We obtain

$$ {}^{m}UM_1(P_n) = \sum_{uv\in E(P_n)} \frac{1}{d_{up}(u)+d_{up}(v)} $$
$$ = \frac{2}{(n-2)+(n-3)} + \frac{(n-3)}{(n-3)+(n-3)} $$
$$ = \frac{2}{2n-5} + \frac{1}{2}. $$

3. Results for Wheel Graphs

Let W_n be a wheel with n+1vertices and $2n$ edges, n□4. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{uv$ □□$E(W_n) \mid d_{up}(u) = 0,\ d_{up}(v) = n\}$, $|E_1| = n$.
$E_2 = \{uv$ □□$E(W_n) \mid d_{up}(u) = d_{up}(v) = n\}$, $|E_2| = n$.

Theorem 1. Let W_n be a wheel with n+1vertices and $2n$ edges, n□4. Then

$$ UGA(W_n) = n. $$

Proof. We deduce

$$GAU(W_n) = \sum_{uv \in E(W_n)} \frac{2\sqrt{d_{up}(u)d_{up}(v)}}{d_{up}(u) + d_{up}(v)}$$

$$= n\frac{2\sqrt{0 \times n}}{0 + n} + n\frac{2\sqrt{n \times n}}{n + n}$$

$$= n.$$

Theorem 2. Let W_n be a wheel with n+1vertices and 2n edges, n□4. Then

$${}^{m}UM_1(W_n) = \frac{3}{2}.$$

Proof. We obtain

$${}^{m}UM_1(W_n) = \sum_{uv \in E(W_n)} \frac{1}{d_{up}(u) + d_{up}(v)}$$

$$= \frac{n}{0 + n} + \frac{n}{n + n}$$

$$= \frac{3}{2}.$$

4. Results for Gear Graphs

A bipartite wheel graph is a graph obtained from W_n with n+1 vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

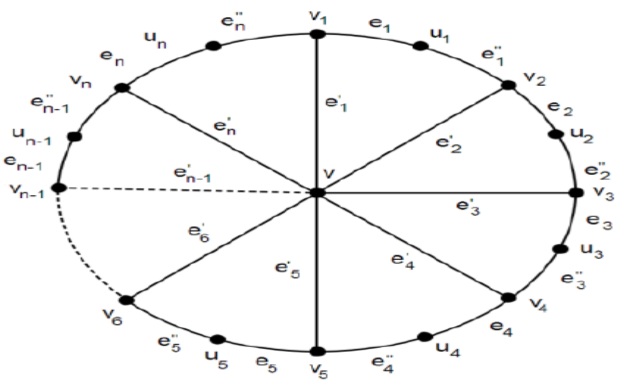

Figure 1. Gear graph G_n

Let G_n be a gear graph with 3n edges, n□4. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

E_1 = {u □□$E(G_n)$ | $d_{up}(u)$ =1, $d_{up}(v)$ = 0}, | E_1 | = n.

E_2 = {u □□$E(G_n)$ | $d_{up}(u)$ = 1, , $d_{up}(v)$ = 3}, | E_2 | = 2n.

Theorem 3. Let G_n be a gear graph with $2n+1$vertices, n□3. Then

$$GAU(G_n) = \sqrt{3}n..$$

Proof: We deduce

$$GAU(G_n) = \sum_{uv \in E(G_n)} \frac{2\sqrt{d_{up}(u)d_{up}(v)}}{d_{up}(u)+d_{up}(v)}$$

$$= n\frac{2\sqrt{1\times 0}}{1+0} + 2n\frac{2\sqrt{1\times 3}}{1+3}$$

$$= \sqrt{3}n.$$

Theorem 4. Let G_n be a gear graph with $2n+1$vertices, n□3. Then

$$^{m}UM_1(G_n) = \frac{3n}{2}.$$

Proof: We deduce

$$^{m}UM_1(G_n) = \sum_{uv \in E(G_n)} \frac{1}{d_{up}(u)+d_{up}(v)}$$

$$= \frac{n}{1+0} + \frac{2n}{1+3}$$

$$= \frac{3n}{2}.$$

5. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

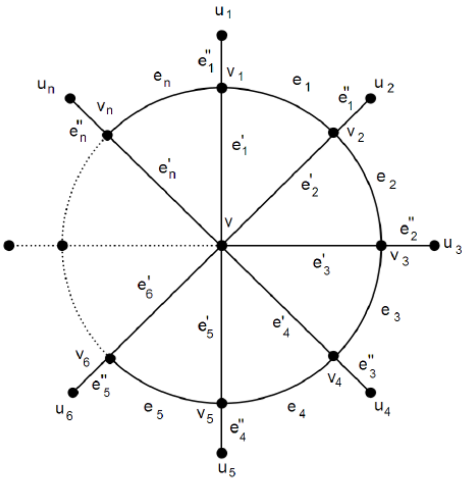

Figure 2. Helm graph H_n

Let H_n be a helm graph with $3n$ edges, n□3. Then H_n has three types of the uphill degree of edges as follows:

$E_1 = \{uv$ □□$E(H_n) \mid d_{up}(u) = n+1, d_{up}(v) = n\}$. $\quad |E_1| = n$.

$E_2 = \{uv \in E(H_n) \mid d_{up}(u) = d_{up}(v) = n\}$. $\quad |E_2| = n$.

$E_3 = \{uv \in E(H_n) \mid d_{up}(u) = n,\ d_{up}(v) = 0\}$. $\quad |E_3| = n$.

Theorem 5. Let H_n be a helm graph with $2n+1$ vertices, $n \geq 3$. Then

$$GAU(H_n) = \frac{2n\sqrt{n(n+1)}}{2n+1} + n.$$

Proof: We obtain

$$GAU(H_n) = \sum_{uv \in E(H_n)} \frac{2\sqrt{d_{up}(u) d_{up}(v)}}{d_{up}(u) + d_{up}(v)}$$

$$= n\frac{2\sqrt{(n+1)\times n}}{(n+1)+n} + n\frac{2\sqrt{n\times n}}{n+n} + n\frac{2\sqrt{n\times 0}}{n+0}$$

$$= \frac{2n\sqrt{n(n+1)}}{2n+1} + n.$$

Theorem 6. Let H_n be a helm graph with $2n+1$vertices, $n \geq 3$. Then

$$^{m}UM_1(H_n) = \frac{n}{2n+1} + \frac{3}{2}.$$

Proof: We deduce

$$^{m}UM_1(H_n) = \sum_{uv \in E(H_n)} \frac{1}{d_{up}(u) + d_{up}(v)}$$

$$= \frac{n}{(n+1)+n} + \frac{n}{n+n} + \frac{n}{n+0}$$

$$= \frac{n}{2n+1} + \frac{3}{2}.$$

7. **Conclusion**

In this paper, the geometric-arithmetic uphill and modified first uphill indices of some standard graphs, wheel graphs, gear graphs and helm graphs are determined.

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