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# Hereditary Property on Algebraic Structures

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**Abstract:** The present paper deals with the hereditary property on different algebraic structures and give some example of algebraic structures which has hereditary property. I discuss the proof and definition of some algebraic structures whose substructure has the same character of the structure.

**Keywords:** Hereditary property, group, cyclic group, abelian group, Hausdorff space, regular space

## 1. Introduction

We are all aware of the basic concepts of different types groups in group theory and different types ring theory, vector space in linear algebra, first countable space, second countable space, Hausdorff space, $T_0$ −space, $T_1$ −space, $T_2$ −space, regular space, normal space in the study of topology and their properties. We already know about some properties of the substructure of the algebraic structure.

## 2. Definitions

**Definition 2.1 Algebraic Structure:** Algebraic structure is representation of non-set S equipped with one or more binary operations, the binary operations satisfy certain axioms viz. Associativity, commutativity, identity, inverses, meet, join, union, intersections, subset etc. These are some examples of algebraic structure viz. Group, Ring, Field, Vector space, Monoid, Semigroup, subgroup, Module, Lattice, Boolean Algebra, Topological spaces, Graph etc.

**Definition 2.2 Hereditary Property:** Let S be an algebraic structure with a property P if H is a substructure of S has the same property P then this property P is called hereditary property on the structure S. If the substructure H has fails to preserved the property P then it has no hereditary property.

## 3. Theorems

**Theorem 3.1:** Every abelian group preserve the hereditary property.

**Proof:** Let $(G, *)$ be an abelian group then its binary operation $*$ satisfy the property of commutativity i.e. $a * b = b * a, \ \forall a, b \in G$

Now let $(H, *)$ be a subgroup of $(G, *)$ then for all $x, y \in H \subseteq G$

$x * y = y * x$

Hence $(H, *)$ is an abelian group and Abelian group has hereditary property.

**Theorem 3.2:** Cyclic group has hereditary property.

**Proof:** Let $(G,*)$ be a cyclic group then every element of G can be expressed as there exists an element $a \in G$ such that each element of G can be expressed as a power of $a$. i.e. $G =< a >$

Let H be a subgroup of G. If H is trivial subgroup then the H is cyclic. If H is non-trivial subgroup of G then let m be the least positive integer such that $a^m \in H$, we claim $H =< a^m >$

Let $x \in H \Rightarrow x \in G \; then \; x = a^k$

By division algorithm $k = mq + r, \;\; 0 \leq r < m$

$\Rightarrow a^r = a^k a^{-mq} = x(a^m)^{-q} \in H \Rightarrow r = 0$

$\Rightarrow k = mq, \;\; x = a^k (a^m)^q$

All members of H is a power of $a^m$

$H =< a^m >$

Hence H is cyclic group and cyclic group has hereditary property.

**Theorem 3.3:** Commutative ring preserve the hereditary property.

**Proof:** Let $(R, \oplus, \otimes)$ be a commutative ring then $a \otimes b = b \otimes a \; for \; all \; a, b \in R$ and let $(S, \oplus, \otimes)$ is closed under same binary operation and satisfy the property of subring of the commutative ring $(R, \oplus, \otimes)$.

Let $x, y \in S \subseteq R$ then $x \otimes y = y \otimes x \;\; for \; all \; x, y \in S$

Hence $(S, \oplus, \otimes)$ is a commutative ring and commutative ring $(R, \oplus, \otimes)$ has hereditary property.

**Theorem 3.4:** Finite dimensional vector space V over the field F preserve the hereditary property.

**Proof:** Let V be a finite dimensional vector over the field F and W be its subspace then we suppose $S_e$ is a linearly independent subset of W. If S is a linearly independent subset of W containing $S_e$ then S is also linearly independent subset of V. Thus S contains no more than the numbers elements of dimension of V.

We extend $S_e$ to a basis for W. If $S_e$ spans W then $S_e$ is a basis for W and we get W is a finite dimensional vector space. If $S_e$ does not span W then we find a vector $\alpha_1$ in W such that the set $S_1 = S_e \cup \{\alpha_1\}$ is linearly independent and we get W is a finite dimensional vector space. If $S_1$ does not span W then we continue this process to find $S_m = S_e \cup \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ which is a basis for W. Thus there is a basis of W containing $\alpha$ which contains no more than dim V elements. Thus W is a finite dimensional vector space. Hence finite dimensional vector space preserve the hereditary property.

**Theorem 3.5:** First countable space has hereditary property.

**Proof:** Let $(X, \tau)$ be a first countable space and let $(Y, \tau_1)$ is a subspace of $(X, \tau)$. Now let y be any arbitrary point of Y than $y \in X$. There exists a countable $\tau-$local base $B(y)$ at y. The collection

$B_1(y) = \{Y \cap B : B \in B(y)\}$ forms a countable $\tau_1$ −local base at y. Thus $(Y, \tau_1)$ is a first countable space. Hence First countable space has hereditary property.

**Theorem 3.6:** Second countable space is the hereditary property

**Proof:** Let $(X, \tau)$ be a second countable with countable base B and let $(Y, \tau_Y)$ is a subspace of $(X, \tau)$then $B_Y = \{B \cap Y : B \in B\}$ is a base for $\tau_Y$.

As we know $B_Y$ is countable. Thus there exists a countable base for $\tau_Y$. Finally we see that subspace of second countable space is second countable space. Hence Second countable space has hereditary property.

**Theorem 3.7:** The $T_0$ –space has the hereditary property.

**Proof:** A space $(X, \tau)$ is said to be $T_0$ –space if and only if for given any pair of distinct points of $X$ there exists a neighbourhood N of one of them not containing other.

Let $(X, \tau)$ is a $T_0$ –space and let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. Let $x$ *and* $y$ be two distinct points of Y then $x$ *and* $y$ are also two distinct points of X then there exists $\tau - open$ neighbourhood $N_x$ of x such that $y \in N_x$

Then $N_x \cap Y$ is a $\tau_Y - open$ neighbourhood of x not containing y.

Hence $(Y, \tau_Y)$ is a $T_0 - space$ and $T_0 - space$ $(X, \tau)$ has hereditary property.

**Theorem 3.8:** The $T_1$ –space has the hereditary property.

**Proof:** A space $(X, \tau)$ is said to be $T_1$ –space if and only if for given any pair of distinct points $x$ and $y$ of $X$ there exist two open sets $O_1$ *and* $O_2$ such that

$x \in O_1 \; but \; y \notin O_1 \;\; and \; y \in O_2 \; but \; x \notin O_2$

Let $(X, \tau)$ is a $T_1$ –space and let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. Let $x$ *and* $y$ be two distinct points of Y then $x$ *and* $y$ are also two distinct points of X then there exist two open sets $O_1$ *and* $O_2$ such that

$x \in O_1 \; but \; y \notin O_1 \;\; and \; y \in O_2 \; but \; x \notin O_2$

Then $G_Y = O_1 \cap Y \;\; and \;\; H_Y = O_1 \cap Y$ are $\tau_Y$ −open sets such that

$x \in G_Y \; but \; y \notin G_Y \;\; and \; y \in H_Y \; but \; x \notin H_Y$

Thus $(Y, \tau_Y)$ is a $T_1$ –space.

Hence $T_1$ –space has hereditary property.

**Theorem 3.9:** Hausdorff space has hereditary property.

**Proof:** A space $(X, \tau)$ is said to be Hausdorff space if and only if for every pair of distinct points x and y of X, there exist disjoint neighbourhoods $N_x \; and \; N_y \; of \; x \; and \; y$ respectively such that $N_x \cap N_y = \emptyset$

Let $(X, \tau)$ is a Hausdorff space and let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. Let $x \: and \: y$ be two distinct points of Y then $x \: and \: y$ are also two distinct points of X then there exist two open neighbourhood $N_x \: and \: N_y \: of \: x \: and \: y$ respectively. Now $N_x \cap Y \: and \: N_y \cap Y$ are $\tau_Y - open \: sets.$

$x \in N_x \: and \: x \in Y \Rightarrow x \in N_x \cap Y$
and $y \in N_y \: and \: y \in Y \Rightarrow y \in N_y \cap Y$ and $N_x \cap N_y = \emptyset$

Thus $(Y \cap N_x) \cap (Y \cap N_y) = Y \cap (N_x \cap N_y) = Y \cap \emptyset = \emptyset$

We see that $N_x \cap Y \: and \: N_x \cap Y$ are disjoint $\tau_Y - open$ neighbourhoods of x and y respectively.

Hence $(Y, \tau_Y)$ is a Hausdorff space and Hausdorff space $(X, \tau)$ has hereditary property.

**Theorem 3.10:** Every regular space has hereditary property.

**Proof:** A space $(X, \tau)$ is said to be regular space if and only if for every $\tau - closed$ set M and every point $x \notin M$, there exist $\tau - open$ sets $O_M \: and \: O_x$ such that

$M \subset O_M, x \in O_x \: and \: O_M \cap O_x = \emptyset$

Let $(X, \tau)$ is a regular space and let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. Let M $\tau_Y - closed$ subset of Y and $x \notin M$ be a point of Y then

$Cl_Y(M) = Cl_X(M) \cap Y$

We have $Cl_Y(M) = M \: as \: M = Cl_X(M) \cap X$

Thus $x \notin M \Rightarrow x \notin Cl_X(M) \cap Y$

$\Rightarrow x \notin Cl_X(M)$

$Cl_X(M) \: is \: a \: \tau - closed$ subset of X such that $x \notin Cl_X(M)$

There exist $\tau - open$ sets $O_M \: and \: O_x$ such that

$Cl_X(M) \subset O_M, x \in O_x \: and \: O_M \cap O_x = \emptyset$

Now $x \in O_x \: and \: x \in Y \Rightarrow x \in O_x \cap Y$

$Cl_X(M) \subset O_M \Rightarrow Cl_X(M) \cap Y \subset O_M \cap Y \Rightarrow M \subset O_M \cap Y$ and $O_x \cap Y \: and \: O_M \cap Y$ are $\tau_Y$ −open subsets of Y such that

$x \in O_x \cap Y \: and \: M \subset O_M \cap Y \: and \: (O_x \cap Y) \cap (O_M \cap Y) = \emptyset$

Thus $(Y, \tau_Y)$ is a regular space and regular space $(X, \tau)$ has hereditary property.

4. **Corollary:** $T_3$ −space has hereditary property since $T_3$ −space is a regular space as well as $T_1$ −space.
5. **Conclusion:** The above study leads to conclude that some algebraic structures have hereditary property some algebraic structures need not have hereditary property. There are some algebraic

structures viz. Non-abelian group, normal space, compact space etc. Does not preserve hereditary property.

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