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Study of Solve Differential Equations using Laplace Transform

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Abstract:

The Laplace transform is a powerful tool for solving differential equations. This method involves transforming a differential equation into an algebraic equation, solving for the transform, and then inverting the transform to obtain the solution. The Laplace transform is particularly useful for solving linear differential equations with constant coefficients, and it can handle discontinuous functions and impulsive inputs.

Keyword: Laplace Transform, approach differential equations, Handles discontinuous, Functions impulsive inputs.

1. Introduction:

The Laplace transform is a powerful tool for solving differential equations. It transforms a differential equation into an algebraic equation, which can be solved more easily Take the Laplace Transform Apply the Laplace transform to the differential equation, using the formula $$ \mathcal {L}\ {f (t)\} = \int_{0}^{\infty} e^ {-st} f (t) dt $$.Simplify the Equation: Simplify the resulting algebraic equation, using properties of the Laplace transform, such as linearity and shifting. Solve for the Transform Solve the algebraic equation for the Laplace transform of the unknown function. Common Laplace Transforms. Unit Step Function $\mathcal {L}\{u(t)\} = \frac{1}{s}$. Exponential Function $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ Sine and Cosine Functions: $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$, $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$ The Laplace transform is a powerful tool for solving differential equations, and its applications are diverse and widespread. Explore more about Laplace transforms or differential equation. Differential equations are mathematical equations that describe the relationship between a function and its derivatives. Solving differential equations is a crucial skill in various fields, including physics, engineering, Physics and Engineering Differential equations describe the laws of motion, heat transfer, and wave propagation. Differential equations. model various real-world phenomena, such as population growth, chemical reactions, and electrical circuits. Laplace transform to solve ordinary differential equations (ODE) are presented. One of the main advantages in using Laplace transform to solve differential equations is that the Laplace transform converts a differential equation into an algebraic equation. Heavy calculations involving decomposition into partial fractions are presented in the appendix.

Example : Use Laplace transform to solve the differential equation \[- 2 y' + y = 0 \] with the initial conditions \(y(0) = 1 \) and \(y \) is a function of time \(t \).

Solution :Let \(Y(s) \) be the Laplace transform of \(y(t) \)
Take the Laplace transform of both sides of the given differential equation: \(\mathscr{L}\{ y(t) \} = Y(s) \)
\(\mathscr{L}\{ -2 y' + y\} = \mathscr{L}\{0 \} \)
Use linearity property of Laplace transform to rewrite the equation as
\(- 2 \mathscr{L}\{ y'\} + \mathscr{L}\{ y\} = \mathscr{L}\{0 \} \)
Use derivative property to rewrite the term \(\mathscr{L}\{ y'\} = (s Y(s) - y(0)) \).
\(- 2 (s Y(s) - y(0)) + Y(s) = 0 \)
Expand the above as
\(- 2 s Y(s) + 2 y(0) + Y(s) = 0 \)
Substitute \(y(0) \) by its given numerical value
\(- 2 s Y(s) + 2 + Y(s) = 0 \)
Solve the above for \(Y(s) \)
\(Y(s) (1 - 2 s) = -2 \)
\(Y(s) = \dfrac{2}{2 s - 1} \)
\(Y(s) = \dfrac{1}{ s - 1/2} \)
We now use formula (3) in the table of formulas of Laplace transform to find the inverse Laplace transform of \(Y(s) \) obtained above as
\(\display style y(t) = e^{\frac{1}{2} t } \)
let's check that the solution obtained \(y(t) = e^{\frac{1}{2} t } \) satisfies the given \(- 2 y' + y = - 2 ((1/2) e^{\frac{1}{2} t }) + e^{\frac{1}{2} t } \)
Simplify the above\(- e^{\frac{1}{2} t } + e^{\frac{1}{2} t } = 0 \) ; differential equation satisfied.
\(y(0) = e^{\frac{1}{2} 0 } = e^0 = 1 \) ; initial value .

Example: Use Laplace transform to solve the differential equation \[y'' - 2 y' -3 y = 0 \] with the initial conditions \(y(0) = 2 \) and \(y'(0) = - 1 \) and \(y \) is a function of time \(t \).
Solution: Let \(Y(s) \) be the Laplace transform of \(y(t) \)
Take the Laplace transform of both sides of the given differential equation
\(\mathscr{L}\{ y'' - 2 y' -3 y \} = \mathscr{L}\{0 \} \)
Use linearity property of Laplace transform to rewrite the equation as
\(\mathscr{L}\{ y''\} - 2 \mathscr{L}\{ y'\} - 3 \mathscr{L}\{ y \} = \mathscr{L}\{0 \} \)
Use first and second derivative properties to rewrite the terms \(\mathscr{L}\{ y''\} \) and \(\mathscr{L}\{ y'\} \) and simplify the right side.

$s^2 Y(s) - s y(0) - y'(0) - 2 (sY(s) - y(0)) - 3 Y(s) = 0$
Substitute $y(0)$ and $y'(0)$ by their numerical values and expand
$s^2 Y(s) - 2 s + 1 - 2 s Y(s) + 4 - 3Y(s) = 0$
Group like terms and keep terms with $Y(s)$ on the left
$s^2 Y(s) - 2 s Y(s) - 3 Y(s) = 2 s - 5$
Factor $Y(s)$ out
$Y(s) (s^2 - 2 s - 3) = 2 s - 5$
Solve the above for $Y(s)$
$Y(s) = \dfrac{2s - 5}{s^2 - 2 s - 3}$
Expand the right side into partial fractions $Y(s) = \dfrac{7}{4\left(s+1\right)}+\dfrac{1}{4\left(s-3\right)}$We now use formula (3) in the table of formulas of Laplace transform to find the inverse Laplace transform of $Y(s)$ which is given by
$\displaystyle y(t) = \dfrac{7}{4} e^{- t } + \dfrac{1}{4} e^{3 t }$
You may check that the solution obtained satisfies the differential equation and the initial values given.

Example: Use Laplace transform to solve the differential equation $$y'' + 2 y' + 2 y = 0$$ with the initial conditions $y(0) = -1$ and $y'(0) = 2$ and y is a function of time t.

Solution : Let $Y(s)$ be the Laplace transform of $y(t)$
Take the Laplace transform of both sides of the given differential equation
$\mathscr{L}\{ y'' + 2 y' + 2 y \} = \mathscr{L}\{0 \}$
Laplace transform to rewrite the equation $\mathscr{L}\{ y''\} + 2 \mathscr{L}\{ y'\} + 2 \mathscr{L}\{ y \} = \mathscr{L}\{0 \}$
Use first and second derivative properties to rewrite the terms $\mathscr{L}\{ y''\}$ and $\mathscr{L}\{ y'\}$ and simplify the right side.
$s^2 Y(s) - s y(0) - y'(0) + 2 (sY(s) - y(0)) + 2 Y(s) = 0$
Substitute $y(0)$ and $y'(0)$ by their numerical values and expand
$s^2 Y(s) + s - 2 + 2 s Y(s) + 2 + 2 Y(s) = 0$
Group like terms and keep terms with $Y(s)$ on the left side of the equation
$s^2 Y(s) + 2 s Y(s) + 2 Y(s) = - s$
Factor $Y(s)$ out
$Y(s) (s^2 + 2 s + 2) = - s$
Solve the above for $Y(s)$
$Y(s) = \dfrac{-s}{s^2 + 2 s + 2}$
Factor denominator over the complex numbers by first solving the equation
$s^2 + 2 s + 2 = 0$
which gives two complex solutions
$S_1 = -1 + j$ and $s_2 = -1 - j$
Factor
$Y(s) = \dfrac{-s}{(s - s_1)(s - s_2)}$

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Expand the right side of the above into partial fractions \dfrac{-s}{(s - s_1)(s - s_2)} = \dfrac{A}{s-s_1} + \dfrac{B}{s-s_2} \)
\(A = \dfrac{-s_1}{s_1-s_2} = \dfrac{-(-1 + j)}{2 j} = -\dfrac{1}{2} - \dfrac{1}{2} j \)
\(B = \dfrac{-s_2}{s_2-s_1} = \dfrac{-(-1 - j)}{-2 j} = - \dfrac{1}{2} + \dfrac{1}{2} j \)
Use formulas in the table of formulas to find the inverse Laplace transform of \(Y(s) = \dfrac{A}{s-s_1} + \dfrac{B}{s-s_2} \) which is given by
\(y(t) = A e^{s_1 t} + B e^{s_2 t} \)
Let us write \(A \) and \(B \) in exponential form
\(A = -\dfrac{1}{2} - \dfrac{1}{2} j = \frac{\sqrt 2}{2} e^{ \frac{-3\pi}{4} j} \)
\(B = -\dfrac{1}{2} + \dfrac{1}{2} j = \frac{\sqrt 2}{2} e^{ \frac{3\pi}{4} j} \)
Substitute \(s_1 \), \(s_2 \), \(A \) and \(B \) by their values and rewrite \(y(t) \) as
\(y(t) = (\frac{\sqrt 2}{2} e^{ \frac{-3\pi}{4} j}) e^{(-1 + j) t} + (\frac{\sqrt 2}{2} e^{ \frac{3\pi}{4} j}) e^{(-1 - j) t} \)
Factor \(\dfrac{\sqrt 2}{2} e^{-t} \) out and group exponents
\(y(t) = \dfrac{\sqrt 2}{2} e^{-t} \left[e^{j t - \frac{3\pi}{4} j } + e^{-j t + \frac{3\pi}{4} j } \right] \)
Use Euler formula (\(e^jx = \cos x + j \sin x \)) to simplify the terms inside the brackets
\(y(t) = \dfrac{\sqrt 2}{2} e^{-t} \left[\cos(t - \frac{3\pi}{4}) + j\sin(t - \frac{3\pi}{4}) + \cos(-t + \frac{3\pi}{4}) + j\sin(- t + \frac{3\pi}{4}) \right] \)
which simplifies to
\(y(t) = \sqrt 2 e^{-t} \cos(t - \frac{3\pi}{4}) \)
You may check that the solution obtained satisfies the differential equation and the initial

Example: Use Laplace transform to solve the differential equation \[y'' - y' - 2 y = \sin (3t) \] with the initial conditions \(y (0) = 1 \) and \ (y' (0) = -1 \).Values given.

Solution : Let \(Y(s) \) be the Laplace transform of \(y(t) \)
Laplace transform of both sides of the given differential equation
\(\mathscr{L}\{ y'' - y' - 2 y \} = \mathscr{L}\{ \sin(3t) \} \)
Laplace transform to expand the left side and use table to evaluate the right side.
\(\mathscr{L}\{ y''\} - \mathscr{L}\{ y'\} - 2 \mathscr{L}\{ y \} = \dfrac{3}{s^2+3^2} \)Use first and second derivative properties to rewrite the terms \(\mathscr{L}\{ y''\} \) and \(\mathscr{L}\{ y'\} \) and simplify the right side. \(s^2 Y(s) - s y(0) - y'(0) - (sY(s) - y(0)) + 2 Y(s) = \dfrac{3}{s^2+3^2} \)
Substitute \(y(0) \) and \(y'(0) \) by their numerical values and expand
\(s^2 Y(s) - s + 1 - s Y(s) + 1 - 2 Y(s) = \dfrac{3}{s^2+3^2} \)
Group like terms and keep terms with \(Y(s) \) on the left side of the equation
\(s^2 Y(s) - s Y(s) - 2 Y(s) = \dfrac{3}{s^2+3^2} + s - 2 \)
Factor \(Y(s) \) out on the left side
\(Y(s) (s^2 - s - 2) = \dfrac{3}{s^2+3^2} + s - 2 \)
Solve the above for \(Y(s) \)
\(Y(s) = \dfrac{3}{(s^2+3^2)(s^2 - s - 2)} + \dfrac{s-2}{s^2 - s - 2} \)

Factor the term $s^2 - s - 2$ in the denominator

$Y(s) = \dfrac{3}{(s^2+3^2)(s-2)(s+1)} + \dfrac{s-2}{(s-2)(s+1)}$

which may be expanded in partial fractions $Y(s) = \dfrac{3s}{130(s^2+3^2)} - \dfrac{33}{130(s^2+3^2)} + \dfrac{9}{10(s+1)} + \dfrac{1}{13(s-2)}$

We now use formulas in the table of formulas of Laplace transform to find the inverse Laplace transform of $Y(s)$ which is given by

$y(t) = \dfrac{3}{130} \cos(3x) - \dfrac{11}{130} \sin(3x) + \dfrac{9}{10} e^{-x} +\dfrac{1}{13} e^{2x}$

Appendix I: Partial fractions decomposition of example 2

Factor denominator

$\dfrac{2s - 5}{s^2 - 2 s - 3} = \dfrac{2s - 5}{(s-3)(s+1)}$

Expand into partial fractions

$\dfrac{2s - 5}{s^2 - 2 s - 3} = \dfrac{A}{s+1} + \dfrac{B}{s-3}$

Multiply all terms of the above by $(s-3)(s+1)$ and simplify

$2s - 5 = A(s-3) + B(s+1)$ (1)

Set $s = 3$ in equation (1)

2(3) - 5 = A(3 -3) + B(3+1)

Simplify and solve for B

$B = 1/4$

Set $s = - 1$ in equation (1) $2(-1) - 5 = A(-1-3) + B(-1+1)$

Simplify and solve for A $A = \dfrac{7}{4}$.

Appendix II : Partial fractions decomposition of example 3

Partial fractions decomposition of $\dfrac{-s}{(s - s_1)(s - s_2)}$

$\dfrac{-s}{(s - s_1)(s - s_2)} = \dfrac{A}{s-s_1} + \dfrac{B}{s-s_2}$

Multiply all terms of the above by $(s - s_1)(s - s_2)$ and simplify

$- s = A (s-s_2) + B(s - s_1)$ (1)

Evaluate the above at $s=s_1$

$- s_1 = A (s_1-s_2) + B(s_1 - s_1)$

Simplify

$-s_1 = A (s_1-s_2)$

Solve for A

$A = \dfrac{-s_1}{s_1-s_2} = \dfrac{-(-1 + j)}{2 j} = -\dfrac{1}{2} - \dfrac{1}{2} j$

Evaluate both sides of equation (1) at $S = s_2$ and find B in a similar way as finding A above

$B = \dfrac{-s_2}{s_2-s_1} = \dfrac{-(-1 - j)}{-2 j} = - \dfrac{1}{2} + \dfrac{1}{2} j$

Appendix III C : Expand in partial fractions from example 4

$\dfrac{3}{(s^2+3^2)(s^2 - s - 2)} + \dfrac{s-2}{s^2 - s - 2}$

Factor denominators

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$\dfrac{3}{(s^2+3^2)(s-2)(s+1)} + \dfrac{s-2}{(s-2)(s+1)}$
Simplify the term on the right
$\dfrac{3}{(s^2+3^2)(s-2)(s+1)} + \dfrac{1}{s+1}$
Express into partial fractions
$\dfrac{3}{(s^2+3^2)(s-2)(s+1)} + \dfrac{1}{s+1} = \dfrac{As + B}{s^2+3^2} + \dfrac{C}{s+1} + \dfrac{D}{s-2}$
Multiply all terms of the above by the denominator $(s^2+3^2)(s-2)(s+1)$ and simplify
$3 + (s^2+3^2)(s-2) = (As + B)(s-2)(s+1) + C (s^2+3^2)(s-2) + D (s^2+3^2)(s+1)$ (1)Select values of s that simplify calculations for the coefficients A, B, C and D Set $s = 2$ on both sides of equation (1)
$3 + (2^2+3^2)(2-2) = (2 A + B)(2-2)(s+1) + C (2^2+3^2)(2-2) + D (2^2+3^2)(2+1)$
Simplify
$3 = 39 D$
Solve for D
$D = \dfrac{1}{13}$
Set $s = -1$ on both sides of equation (1)
$3 + ((-1)^2+3^2)(-1-2) = (-A + B)(-1-2)(-1+1) + C ((-1)^2+3^2)(-1-2) + D ((-1)^2+3^2)(-1+1)$
Simplify
$3 - 30 = - 30 C$
Solve for C
$C = \dfrac{9}{10}$
Set $s = 0$ on both sides of equation (1)
$3 +(0^2+3^2)(0-2) = (0 + B)(0-2)(0+1) + C (0^2+3^2)(0-2) + D (0^2+3^2)(0+1)$
Simplify
$3 - 18 = -2 B - 19 C + 9D$
Substitute C and D by their numerical values obtained above and solve for B to obtain
$B = -\dfrac{33}{130}$
Set $s = 1$ on both sides of equation (1)
$3 + (1^2+3^2)(1-2) = (A + B)(1-2)(1+1) + C (1^2+3^2)(1-2) + D (1^2+3^2)(1+1)$
Substitute B, C and D by their numerical values obtained above and solve for A to obtain
$A = \dfrac{3}{130}$
Hence
$\dfrac{3}{(s^2+3^2)(s^2 - s - 2)} + \dfrac{s-2}{s^2 - s - 2}$
$\quad \quad = \dfrac{As}{s^2+3^2} + \dfrac{B}{s^2+3^2} + \dfrac{C}{s+1} + \dfrac{D}{s-2}$
$\quad \quad = \dfrac{ 3s}{130(s^2+3^2)} - \dfrac{33}{130(s^2+3^2)} + \dfrac{9}{10(s+1)} + \dfrac{1}{13(s-2)}$.

Conclusion: Using Laplace transforms to solve differential equations provides a powerful and efficient method for finding solutions. By transforming the differential equation into an algebraic equation, we can simplify the problem and solve for the Laplace transform of the unknown function. Then, by taking the inverse Laplace transform, we can obtain the solution to the original differential equation. The Laplace transform is a valuable tool for solving differential equations, and its applications are diverse and widespread. By mastering the Laplace transform method, you can efficiently solve a wide range of differential equations and tackle complex problems in various fields.

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