International Journal on Science and Technology (IJSAT)

E-ISSN: 2229-7677 ● Website: www.ijsat.org ● Email: editor@ijsat.org

Existence and Uniqueness Results for Second-Order Random Differential Equations via Branciari-type Two-Operator Fixed Point Theorem

P. D. Bhosale[1] and **S. S. Bellale**[2], **V. V. Latpate**[3] and **S. V. Badgire**[4]

Research Scholar, Mathematics Research Centre, Dayanand Science College, Latur[1],
Head, Department of Mathematics , Dayanand Science College, Latur[2],
Head, Department of Mathematics, A. C. S College Gangakhed, Dist: Parbhani[3]
Head, Department of Mathematics, Azad Mahavidyalay, Ausa, Dist: Latur[4],
pavanrajed80@gmail.com, sidhesh.bellale@gmail.com,
Vishnuvlatpate@gmail.com, sanjayvbadgire@gmail.com

Abstract: This paper investigates the existence and uniqueness of solutions for a class of second-order random differential equations using fixed point theory. A transformation into an equivalent random integral equation is provided, followed by operator construction. A concrete example demonstrates the main result. This study contributes to the growing field of random differential equations by offering a rigorous and generalizable framework applicable in various stochastic modelling contexts

Keywords: Random differential equation, fixed point theorem, second-order differential equation, random operator, existence and uniqueness, integral equation, stochastic process.

1. Introduction and Statement of the Problem

Random differential equations (RDEs) have gained substantial importance due to their applicability in diverse disciplines such as physics, biology, engineering, and finance, where systems are influenced by randomness or noise. In practical models, uncertainties in initial conditions, system parameters, or forcing terms are better described by incorporating random variables or stochastic processes into the model formulation. This motivates the need for a theoretical framework to analyze such equations, particularly in terms of existence and uniqueness of solutions.

Traditional deterministic methods fall short when randomness is introduced. A new class of analytical tools involving random fixed point theorems has been developed to prove the solvability of such equations under compactness, continuity, and measurability conditions .

In this paper, we consider the following second-order random differential equation:

$$\frac{d^2x(t,\omega)}{dt^2} = f(t, x(t,\omega), \omega), \quad t \in [a,b],\ \omega \in \Omega, \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

subject to the initial/boundary conditions:

$$x(a,\omega) = x_0(\omega), \quad \frac{dx}{dt}(a,\omega) = x_1(\omega), \ldots\ldots (2)$$

where:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space,
- $x(t,\omega)$ is a real-valued function (random process),
- $f: [a,b] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is jointly measurable and satisfies certain growth and Lipschitz-type conditions .

Our goal is to establish the existence and uniqueness of a random solution $x(t,\omega)$ satisfying the above second-order differential equation using random fixed point theorems in a suitably defined function space.

The contribution of this work lies in:

1. Converting the second-order random differential equation into an equivalent random integral equation;
2. Reformulating it as an operator equation;
3. Applying an appropriate random fixed point theorem to prove existence and uniqueness;
4. Demonstrating the applicability through a concrete example.

2. Preliminaries

Definition 1 (Probability Space). A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

- Ω is a sample space,
- $\mathcal{F}$ is a σ-algebra of subsets of Ω,
- $\mathbb{P}: \mathcal{F} \to [0,1]$ is a probability measure with $\mathbb{P}(\Omega) = 1$.

Definition 2 (Random Variable). A function $X: \Omega \to \mathbb{R}$ is a random variable if it is $\mathcal{F}$-measurable, i.e., for every Borel set $B \subseteq \mathbb{R}$, we have $X^{-1}(B) \in \mathcal{F}$.

Definition 3 (Random Function). A function $f: [a,b] \times \Omega \to \mathbb{R}$ is a random function if:

- for each fixed $t \in [a,b]$, the function $\omega \mapsto f(t,\omega)$ is measurable,
- for each fixed $\omega \in \Omega$, the function $t \mapsto f(t,\omega)$ is continuous.

Definition 4 (Random Operator). Let (X, d) be a metric space. A function $T: X \times \Omega \to X$ is a random operator if:

- for each fixed $x \in X$, the mapping $\omega \mapsto T(x,\omega)$ is measurable,
- for each fixed $\omega \in \Omega$, the mapping $x \mapsto T(x,\omega)$ is continuous.

3. Lemma – Integral and Operator Form

We consider the second-order random differential equation:

$$\frac{d^2x(t,\omega)}{dt^2} = f(t, x(t,\omega), \omega), \quad t \in [a,b] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

with initial conditions:

$$x(a,\omega) = x_0(\omega), \quad \frac{dx}{dt}(a,\omega) = x_1(\omega), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where $f: [a,b] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is jointly measurable and satisfies suitable Lipschitz-type conditions.

The initial value problem [3]–[4] is equivalent to the integral equation:

$$x(t,\omega) = x_0(\omega) + x_1(\omega)(t-a) + \int_a^t \int_a^s f(r, x(r,\omega), \omega)\, dr\, ds.$$

Proof:

Step 1: Integrate equation [3] with respect to t:

$$\frac{dx}{dt}(t,\omega) = x_1(\omega) + \int_a^t f(s, x(s,\omega), \omega)\, ds$$

Step 2: Integrate again:

$$\begin{aligned} x(t,\omega) &= x(a,\omega) + \int_a^t \frac{dx}{dt}(s,\omega)\, ds \\ &= x_0(\omega) + \int_a^t \left[x_1(\omega) + \int_a^s f(r, x(r,\omega), \omega)\, dr \right] ds \\ &= x_0(\omega) + x_1(\omega)(t-a) + \int_a^t \int_a^s f(r, x(r,\omega), \omega)\, dr\, ds \end{aligned}$$

Hence, the solution to [3]–[4] is equivalent to equation [5]. ▫

Operator Formulation

Let $C([a,b] \times \Omega)$ denote the space of all continuous random functions. Define:

Operator A:

$$(Ax)(s,\omega) = \int_a^s f(r, x(r,\omega), \omega)\, dr$$

Operator B:

$$(Bx)(t,\omega) = x_0(\omega) + x_1(\omega)(t-a) + \int_a^t x\,(s,\omega)\,ds$$

Then, the integral equation becomes:

$$x(t,\omega) = (B \circ A)(x)(t,\omega)$$

That is, the solution is a fixed point of the operator $T = B \circ A$, i.e., $T(x) = x$.

4. Theorem–Branciari-type Two-Operator Fixed Point Theorem

Let (X,d) be a complete metric space. Let $A: X \to Y$ and $B: Y \to X$ be two mappings such that the composition $T = B \circ A: X \to X$ is a contraction. That is, there exists a constant $0 < k < 1$ such that:

$$d(Tx,Ty) = d(B(Ax),B(Ay)) \le k \cdot d(x,y) \qquad \text{for all } x,y \in X$$

Then, T has a unique fixed point $x^* \in X$. That is:

$$T(x^*) = x^* \Rightarrow B(A(x^*)) = x^*$$

This implies that the original second-order random differential equation has a unique random solution.(5)

4.1 Hypotheses

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and let $C([a,b]\times\Omega)$ be the space of all real-valued functions $x: [a,b]\times\Omega \to \mathbb{R}$ such that for each $\omega \in \Omega$, the function $t \mapsto x(t,\omega)$ is continuous. Define the metric d on this space as:

$$d(x,y) = \sup_{t\in[a,b]} \mathbb{E}[|x(t,\omega) - y(t,\omega)|]$$

- **Joint Measurability:**
  For each $t \in [a,b]$ and $x \in \mathbb{R}$, the function $\omega \mapsto f(t,x,\omega)$ is measurable. For each $\omega \in \Omega$, the function $(t,x) \mapsto f(t,x,\omega)$ is continuous.

- **Lipschitz Condition:**
  There exists a constant $L > 0$ such that for all $x_1, x_2 \in \mathbb{R},\ t \in [a,b],\ \omega \in \Omega$:

  $$|f(t,x_1,\omega) - f(t,x_2,\omega)| \le L|x_1 - x_2|$$

- **Continuity of Initial Conditions:**
  The initial conditions $x_0(\omega)$ and $x_1(\omega)$ are continuous random variables.

- **Space Completeness:**
  The function space $C([a,b]\times\Omega)$ with metric d is complete.

- **Contraction Condition:**
  Let $T = B \circ A$, and assume $L(b-a)^2/2 < 1$ so that T is a contraction.

4.2 Main Theorem Statement

(Existence and Uniqueness via Two-Operator Fixed Point Theorem):
Let the hypotheses (H1)–(H5) hold. Then, the operator $T = B \circ A$, defined by:

$$T(x)(t,\omega) = x_0(\omega) + x_1(\omega)(t-a) + \int_a^t \int_a^s f\,(r, x(r,\omega), \omega)\, dr\, ds$$

has a unique fixed point in the space $C([a,b] \times \Omega)$. That is, there exists a unique function $x(t,\omega)$ such that:

$$T(x)(t,\omega) = x(t,\omega)$$

This fixed point is the unique random solution to the second-order differential equation:

$$\frac{d^2x(t,\omega)}{dt^2} = f(t, x(t,\omega), \omega),$$

with initial conditions:

$$x(a,\omega) = x_0(\omega), \quad \frac{dx}{dt}(a,\omega) = x_1(\omega)$$

4.3 Proof:

Step 1: Define the Operators
We define the operators as follows:

Operator A:

$$(Ax)(s,\omega) = \int_a^s f\,(r, x(r,\omega), \omega)\, dr$$

Operator B:

$$(Bx)(t,\omega) = x_0(\omega) + x_1(\omega)(t-a) + \int_a^t x\,(s,\omega)\, ds$$

Then, the composite operator $T = B \circ A$ is given by:

$$(Tx)(t,\omega) = x_0(\omega) + x_1(\omega)(t-a) + \int_a^t \int_a^s f\,(r, x(r,\omega), \omega)\, dr\, ds$$

Step 2: Show that T Maps into the Space
We want to show that $T(x) \in C([a,b] \times \Omega)$ for every $x \in C([a,b] \times \Omega)$.

From Hypothesis (H1), f is jointly measurable and continuous in t for fixed ω, and measurable in ω for fixed t. Since $x(t,\omega)$ is continuous in t, it follows that $f(t,x(t,\omega),\omega)$ is integrable on $[a,b]$ for each $\omega \in \Omega$.

Thus:

- The inner integral $\int_a^s f\,(r,x(r,\omega),\omega)\,dr$ is continuous in s,
- The outer integral over s is continuous in t,
- $(Tx)(t,\omega)$ is jointly measurable and continuous in t.

Hence, $T(x) \in C([a,b] \times \Omega)$.

Step 3: Prove that T is a Contraction

Let $x, y \in C([a,b] \times \Omega)$. Then:

$$|T(x)(t,\omega) - T(y)(t,\omega)| = \left| \int_a^t \int_a^s [f(r,x(r,\omega),\omega) - f(r,y(r,\omega),\omega)]\,dr\,ds \right|$$

Using the Lipschitz condition (H2):

$$\leq \int_a^t \int_a^s L\,|x(r,\omega) - y(r,\omega)| dr\,ds$$

$$\leq L \cdot \sup_{r\in[a,b]} |x(r,\omega) - y(r,\omega)| \cdot \int_a^t \int_a^s 1\ dr\,ds = L \cdot \| x - y \|_\infty \cdot \frac{(t-a)^2}{2}$$

Taking supremum over $t \in [a,b]$:

$$\| T(x) - T(y) \| \leq \frac{L(b-a)^2}{2} \cdot \| x - y \|$$

Let $k = \frac{L(b-a)^2}{2}$. By Hypothesis (H5), $k < 1$, so T is a contraction.

Step 4: Apply the Banach Fixed Point Theorem

Since T is a contraction on the complete metric space $(C([a,b] \times \Omega), d)$, by the Banach Fixed Point Theorem, T has a unique fixed point $x^* \in C([a,b] \times \Omega)$ such that:

$$T(x^*)(t,\omega) = x^*(t,\omega)$$

This function $x^*(t,\omega)$ is the unique solution to the original second-order random differential equation:

$$\frac{d^2x(t,\omega)}{dt^2} = f(t,x(t,\omega),\omega),$$

with initial conditions:

$$x(a,\omega) = x_0(\omega), \quad \frac{dx}{dt}(a,\omega) = x_1(\omega)$$

4. Verified Example with Explicit Hypothesis

Example 4.1
Consider the second-order random differential equation:

$$\frac{d^2x(t,\omega)}{dt^2} = -x(t,\omega) + \omega sin(t), \quad t \in [0,1],\ \omega \in \Omega$$

with initial conditions:

$$x(0,\omega) = \omega, \quad \frac{dx}{dt}(0,\omega) = 0$$

Here, ω is a random variable uniformly distributed over $[0,1]$.

Step 1: Explicit Check of Hypotheses (H1 to H5)

H1: Joint Measurability and Continuity

- $f(t,x,\omega) = -x + \omega sin(t)$
- Continuous in t for fixed ω (since $sin(t)$ is continuous)
- Linear in $x \Rightarrow$ continuous in x
- Measurable in $\omega \Rightarrow$ valid by composition rules

H1 is satisfied.

H2: Lipschitz Condition

$$|f(t,x_1,\omega) - f(t,x_2,\omega)| = |-x_1 + x_2| = |x_1 - x_2|$$

Lipschitz constant: $L = 1$
H2 is satisfied.

H3: Initial Conditions

$$x_0(\omega) = \omega, \quad x_1(\omega) = 0$$

ω is uniformly distributed on $[0,1] \Rightarrow$ measurable.
H3 is satisfied.

H4: Function Space Completeness
$C([0,1] \times \Omega)$ with sup-metric is a standard complete space of continuous functions.
H4 is satisfied.

H5: Contraction Condition

$$L = 1, \quad (b-a)^2 = 1$$

$$k = \frac{L(b-a)^2}{2} = \frac{1}{2} = 0.5 < 1$$

H5 is satisfied.

Step 2: Convert to Integral Equation
From Lemma 2.5, we convert the differential equation to an integral form:

$$x(t,\omega) = \omega + \int_0^t \int_0^s [-x(r,\omega) + \omega sin(r)]\, dr\, ds$$

Step 3: Picard Iteration
Let initial guess: $x_0(t,\omega) = \omega$

Then:

$$x_1(t,\omega) = \omega + \int_0^t \int_0^s [-x_0(r,\omega) + \omega sin(r)]\, dr\, ds = \omega + \int_0^t \int_0^s [-\omega + \omega sin(r)]\, dr\, ds$$
$$= \omega + \omega \int_0^t \int_0^s (sin(r) - 1)\, dr\, ds$$

Now:

$$\int_0^s (sin(r) - 1)\, dr = -s + 1 - cos(s)$$

Then:

$$\int_0^t (-s + 1 - cos(s))\, ds = I(t)$$

$$\Rightarrow x_1(t,\omega) = \omega(1 + I(t))$$

Step 4:
All assumptions (H1–H5) are satisfied.
The integral form is valid. The first iteration gives a bounded measurable function.
Since T is a contraction, Picard iteration converges to a unique fixed point.
Therefore, this example is valid and satisfies the main theorem.

5. Conclusion

In this study, we rigorously established existence and uniqueness results for a class of second-order random differential equations using a fixed point approach. We systematically transformed the given differential equation into an equivalent integral form and used a contraction-type operator mapping within an appropriate function space. A key contribution of this work is the use of detailed hypothesis verification through a complete and validated example that adheres to all fixed point criteria. This approach not only guarantees theoretical soundness but also demonstrates its applicability in stochastic dynamic systems. Future research may extend this method to fractional or coupled systems.

References:

1. V. Lakshmikantham and S. Leela, Nonlinear Differential Equations in Abstract Spaces, Pergamon Press, 1981.
2. D. O'Regan and M. Meehan, Existence Theory for Nonlinear Integral and Integro-differential Equations, Kluwer Academic Publishers, 1998.
3. B. C. Dhage, "Generalized Contractions and Applications to Nonlinear Functional Equations", Yokohama Mathematical Journal, **48** (2000), 1–10.
4. J. B. Conway, A Course in Functional Analysis, Springer, 2nd Edition, 2000.
5. M. Branciari, "A Fixed Point Theorem of Banach–Caccioppoli Type on a Class of Generalized Metric Spaces", International Journal of Mathematics and Mathematical Sciences, 2002(11): 645–650.
6. C. Bharucha-Reid, Random Integral Equations, Academic Press, 1972.
7. K. R. Parthasarathy, Probability Measures on Metric Spaces, AMS Chelsea Publishing, 2005.
8. B. Pathak and R. Sharma, "Existence of Random Solutions for Second-Order Differential Equations", Random Operators and Stochastic Equations, 23(1), 2015, 31–42.
9. R. Gupta and V. Kumar, "Random Fixed Point Theorems in Metric Spaces and Applications", Mathematics, 2021, **9**(18), 2240. DOI: 10.3390/math9182240
10. R. S. Yadav and M. Singh, "Existence Results for Random Differential Equations via Random Fixed Point Theorems", International Journal of Nonlinear Analysis and Applications, **13**(2), 2022, 277–289.
11. A.Kumar and S. Radenović, "Fixed Point Results for Hybrid Mappings in Random Metric Spaces with Applications", Symmetry, 2022, **14**(3), 478. DOI: 10.3390/sym14030478
12. M. Abbas and M. Alghamdi, "Existence of Random Fixed Points in Generalized Metric Spaces Using Simulation Functions", AIMS Mathematics, **6**(6), 2021, 6522–6543. DOI: 10.3934/math.2021381
13. S. Choudhary and P. Kumar, "Fixed Points of Weakly Contractive Random Operators and Applications to Stochastic DEs", Journal of Fixed Point Theory and Applications, 2023, **25**(1):12. DOI: 10.1007/s11784-023-01020-1
14. R. P. Agarwal and D. O'Regan, "Infinite Interval Problems for Differential Equations in Banach Spaces", Nonlinear Analysis, **75**(4), 2012, 2024–2033.
15. M. Kostić, Abstract Volterra Integro-Differential Equations: Regularity and Solvability in Banach Spaces, SpringerBriefs in Mathematics, Springer, 2022.