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Unveiling New Insights into the Geometry of the Parabola

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ABSTRACT

This research article explores the geometric and mathematical significance of the parabola; a fundamental conic section defined as the locus of a point whose distance from a fixed point (focus) and a fixed line (directrix) are equal. Moving beyond its classical definition, the study introduces a set of one hundred newly derived theorems concerning the tangent, normal, sub-tangent, sub-normal and focus of the parabola. Each theorem is rigorously developed with detailed proofs, step-by-step derivations, and illustrative diagrams to aid comprehension. In addition, the article presents a further 100 original theorems that establish precise mathematical relationships between the tangent, normal, focus, and other essential elements of the parabola. A preamble provides the fundamental formulas necessary for the derivations. Collectively, these contributions expand the theoretical framework of parabola geometry and serve as a significant reference for scholars engaged in advanced studies of conic sections and related fields.

Keywords: Parabola, Conic sections, Tangent, Normal, Directrix.

1. INTRODUTION

The parabola, one of the most fundamental conic sections, holds a central place in both pure and applied mathematics. Defined as the locus of a point equidistant from a fixed point (the focus) and a fixed line (the directrix), the parabola has been extensively studied for its elegant geometric properties and wide-ranging applications in physics, engineering, and computational sciences. Despite its classical treatment in mathematics, the parabola continues to reveal deeper structural insights when examined through modern approaches.

This research article extends the traditional study of the parabola by introducing newly derived theorems concerning its tangent, normal, and focus. Each theorem is rigorously developed with comprehensive proofs, detailed step-by-step derivations, and carefully constructed diagrams to facilitate understanding. Beyond these results, the article also establishes refined mathematical relationships between the tangent, normal, focus, and other essential geometric elements of the parabola. To support these derivations, a preamble is provided containing the fundamental formulas and parametric equations of the parabola, which serve as the basis for the logical progression of proofs and these contributions enrich the theoretical framework of parabola geometry, offering new perspectives for advanced studies in conic sections and related mathematical fields.

Analysis & Derivations of basic equations

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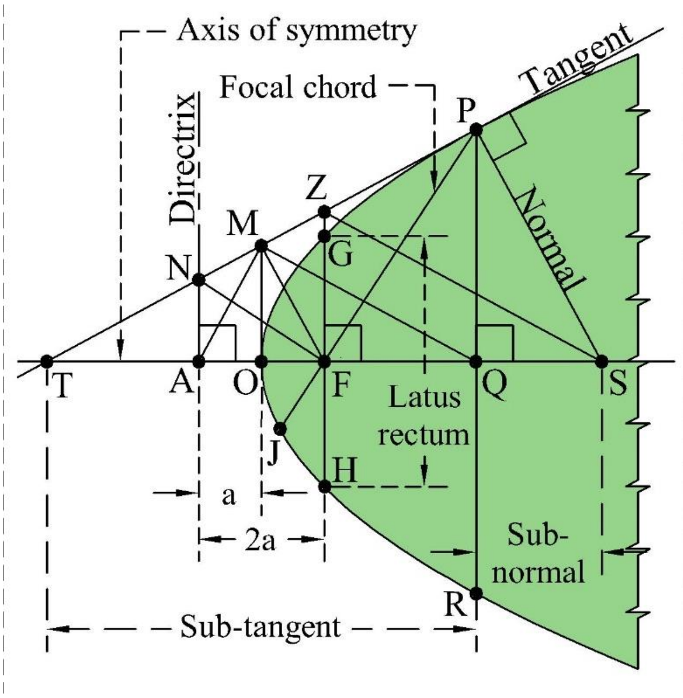

Fig. 1

Common descriptions for the analysis of the elements of a parabola $\boldsymbol{y^2 = 4ax}$ are

Referring fig. 1,

Point O is vertex [1] of parabola,

Point F is known as focus,

point A is the intersection of axis of symmetry [2] and Directrix [3].

Point P is anywhere on the parabola and a tangent is drawn at the point P.

Point Q is projection of point P on axis of symmetry.

Co-ordinates of point P at anywhere on parabola $y^2 = 4ax$ with respect to parametric form is $(\mathrm{at}^2, 2\mathrm{at})$. Therefore, $\mathrm{OQ} = \mathrm{x} = \mathrm{at}^2$ & $\mathrm{PQ} = \mathrm{OR} = \mathrm{y} = 2\mathrm{at}$, where, t is called parameter of the parabola.

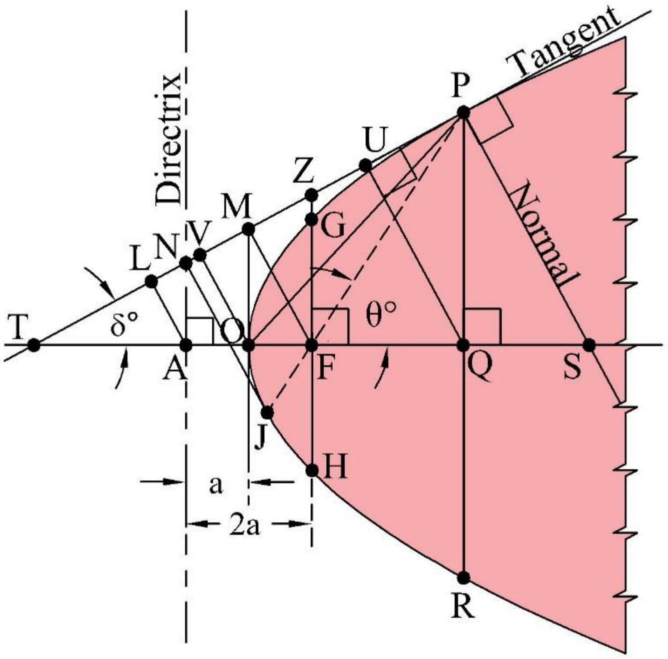

Fig. 2

Point T is intersection of tangent with axis of symmetry.

point S is intersection of normal with axis of symmetry.

Points G & H are extremities of Latus rectum [4] of the parabola and value of GH is equal to 4a.

Points Q is projection of point P on axis of symmetry & tangent at vertex respectively.

Points R chord perpendicular to axis of symmetry through point Q

Point M is intersection of vertical line at vertex O and tangent.

Point N is intersection of Directrix and tangent.

Point J is intersection of focal chord [5] with the parabola.

Referring fig. 2,

Point L is intersection of perpendicular line to tangent through point A.

Point V is intersection of perpendicular line to tangent through point O.

Point U is intersection of perpendicular line to tangent through point Q.

Point U is intersection of tangent and perpendicular line to tangent through point Q.

Point μ is intersection of perpendicular line to tangent through point J and vertical line through vertex O.

Referring fig. 3,

Point E is intersection of perpendicular line to normal from point A.

Point B is intersection of perpendicular line to normal from point O.

Point C is intersection of perpendicular line to normal t from point F.

Point D is intersection of perpendicular line to normal from point Q.

Point K is intersection of perpendicular line to normal from point R.

Point Ω is intersection of perpendicular line to normal from point J

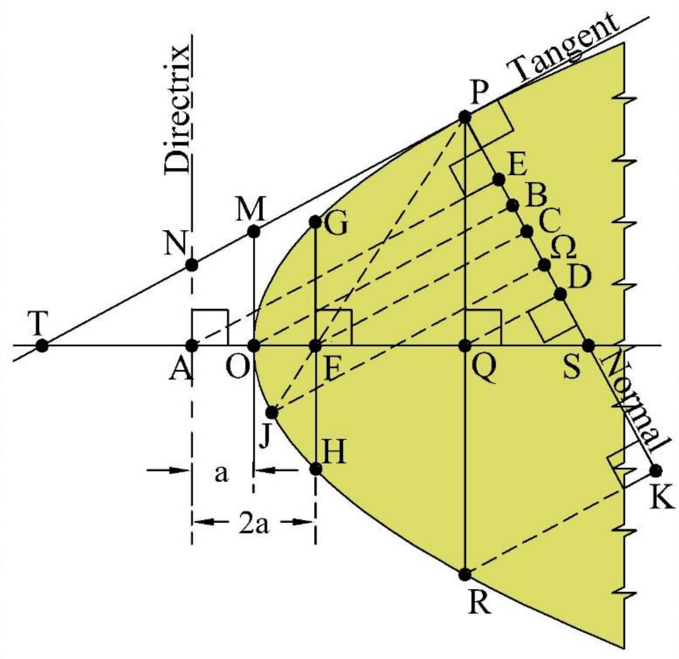

Fig. 3

Referring fig. 4,

$$\angle PFQ = \theta°$$

$$\angle POQ = \varphi°$$

$$\angle PTO = \delta° = \theta°/2$$

$$\text{If} \angle PFQ = \theta°, \qquad \angle TPF = \angle PFC = \angle CFS = \angle QPS = \angle OMF = \theta°/2$$

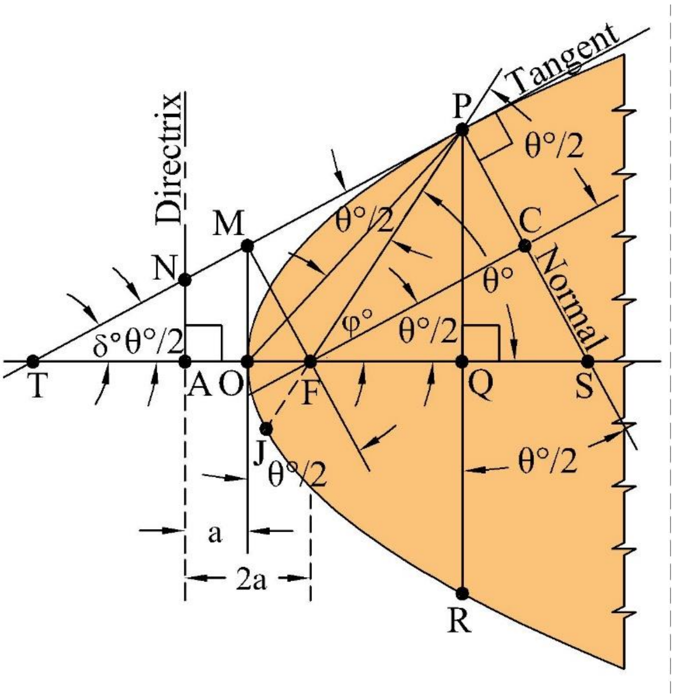

Fig. 4

Referring fig. 5,

Point I is intersection of perpendicular line from point J and the tangent.

Point W is intersection of line JI and axis of symmetry.

Point X is intersection of focal chord through F & M and the parabola.

Point Y is intersection of focal chord through F & M and the parabola.

Point Z is intersection of perpendicular line from point F and the tangent.

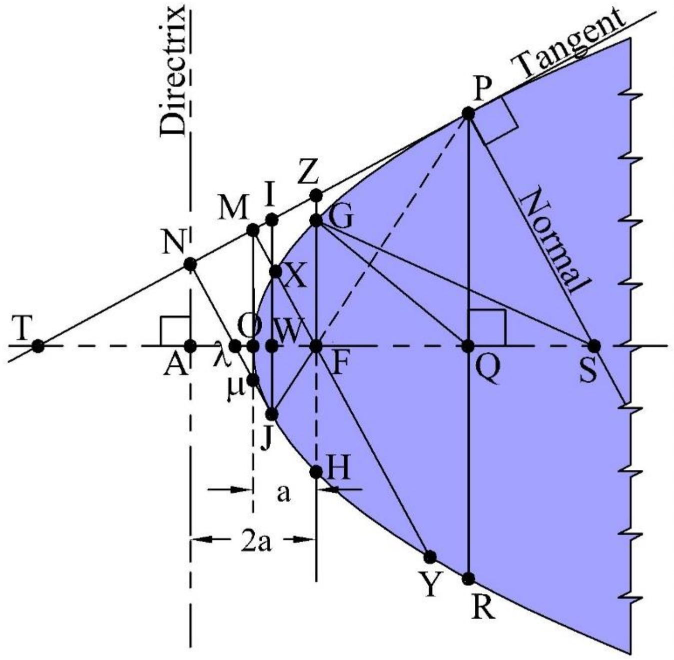

Fig. 5

Referring fig. 6,

Point λ is intersection of line JN and axis of symmetry.

Point μ is intersection of line JN and tangent at vertex.

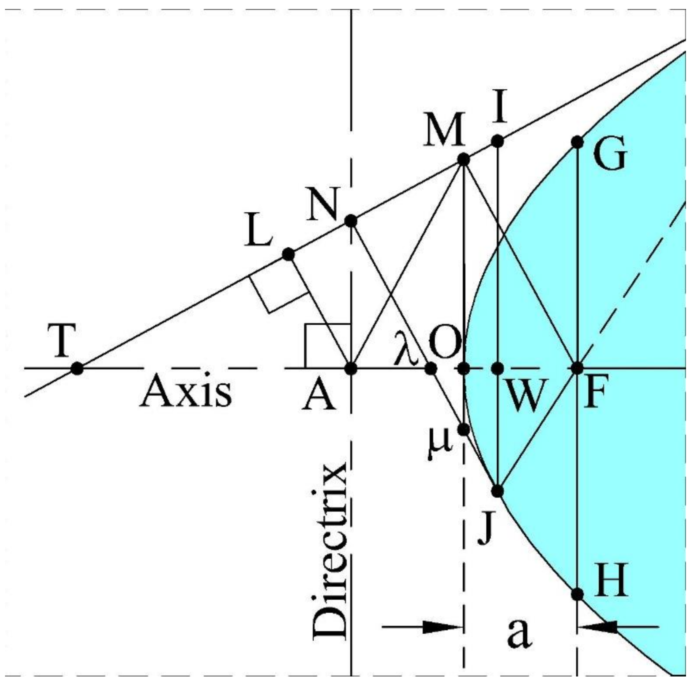

Fig. 6

Referring fig. 7,

Point ∂ is intersection of tangent at vertex and normal at P.

Point τ is intersection of vertical line at F and the normal at P.

Point σ is intersection of tangent at vertex and a line parallel to axis of symmetry through point P.

Point ω is intersection of vertical line at F a line parallel to axis of symmetry through point P.

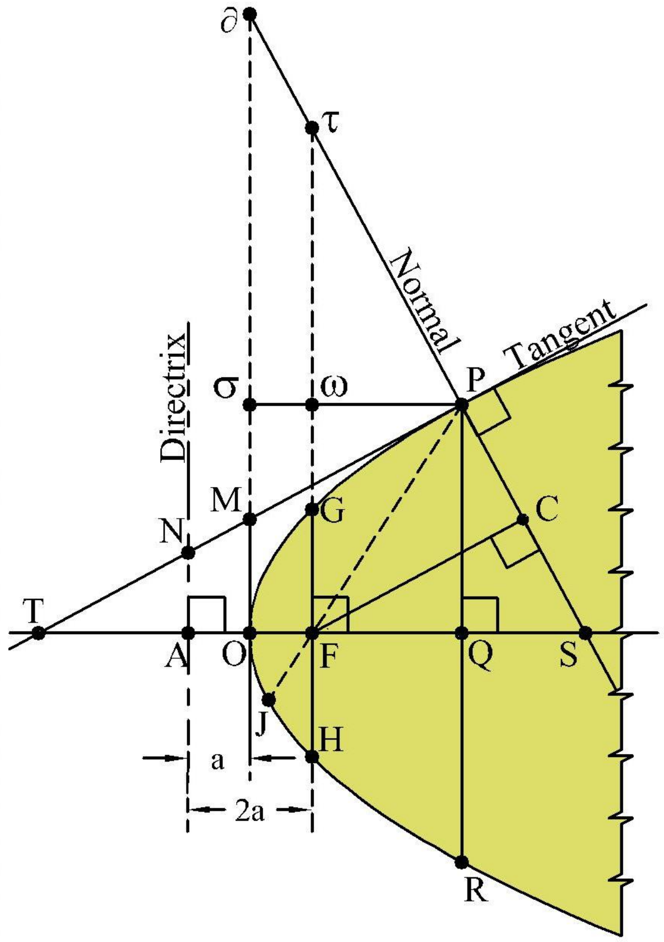

Fig. 7

PREAMBLE- A

Referring fig.1,

Let, Focal length $(OF) = a$, OF is also called as Semi-parameter of the parabola. FA is known as Parameter. FP is called as focal distance [6].

$$OQ = at^2 \quad (A.1)$$

$$OR = PQ = 2at \qquad (A.2)$$

$$AF = 2a \qquad (A.3)$$

$$OF = OA = \frac{AF}{2} = a \qquad (A.4)$$

$$FG = FH = 2a \qquad (A.5)$$

$$SQ = 2a \qquad (A.6)$$

Referring fig.1,

$$FQ = OQ - OF$$

Substituting eqns. (A.1) and (A.4) in above,

$$\therefore FQ = (a \times t^2) - a$$

$$\therefore FQ = a(t^2 - 1) \qquad (A.7)$$

Referring fig.1,

$$OP^2 = OQ^2 + PQ^2$$

Substituting eqns. (A.1) and (A.2) in above,

$$OP^2 = (a \times t^2)^2 + (2at)^2$$

$$\therefore OP^2 = a^2t^4 + 4a^2t^2$$

$$\therefore OP^2 = a^2t^2(t^2 + 4)$$

$$\therefore OP = at\sqrt{t^2 + 4} \qquad (A.8)$$

Referring fig.1,

$$FP^2 = FQ^2 + PQ^2$$

Substituting eqns. (A.7) and (A.2) in above,

$$\therefore FP^2 = [a(t^2 - 1)]^2 + (2at)^2$$

$\therefore FP^2 = a^2(t^2 - 1)^2 + 4a^2t^2$

$\therefore FP^2 = a^2(t^4 + 1 - t^2) + 4a^2t^2$

$\therefore FP^2 = a^2t^4 + a^2 - 2a^2t^2 + 4a^2t^2$

$\therefore FP^2 = a^2t^4 + a^2 + 2a^2t^2$

$\therefore FP^2 = a^2(t^4 + 1 + 2a^2t^2)$

$\therefore FP^2 = a^2(t^2 + 1)^2$

$$\therefore FP = a(t^2 + 1) \quad (A.9)$$

Referring fig.1,

$OS = OQ + QS$

Substituting eqns. (A.1) and (A.6) in above,

$OS = at^2 + 2a$

$$\therefore OS = a(t^2 + 2) \quad (A.10)$$

Referring fig.1, we already know that

$OT = OQ$

$$\therefore OT = at^2 \quad (A.11)$$

Referring fig.1,

$TQ = OT + OQ$

Substituting eqns. (A.11) and (A.1) in above,

$\therefore TQ = at^2 + at^2$

$$\therefore TQ = 2at^2 \quad (A.12)$$

Referring fig.1,

$PT^2 = TQ^2 + PQ^2$

Substituting eqns. (A.12) and (A.2) in above,

$\therefore PT^2 = (2at^2)^2 + (2at)^2$

$\therefore PT^2 = 4a^2t^4 + 4a^2t^2$

$\therefore PT^2 = 4a^2t^4(t^2 + 1)$

$$\therefore PT = 2at\sqrt{t^2 + 1} \qquad (A.13)$$

Referring fig.1, in right-triangle PQS

$PS^2 = QS^2 + PQ^2$

Substituting eqns. (A.6) and (A.2) in above,

$PS^2 = (2a)^2 + (2at)^2$

$\therefore PS^2 = 4a^2 + 4a^2t^2$

$\therefore PS^2 = 4a^2(t^2 + 1)$

$$\therefore PS = 2a\sqrt{t^2 + 1} \qquad (A.14)$$

Referring fig.1,

$TS = TQ + SQ$

Substituting eqns. (A.12) and (A.6) in above,

$TS = 2at^2 + 2a$

$\therefore TS = 2at^2 + 2a$

$$\therefore TS = 2a(t^2 + 1) \qquad (A.15)$$

Referring fig.1,

$TF = OT + OF$

Substituting eqns. (A.11) and (A.4) in above,

$\therefore \ TF = at^2 + a$

$$\therefore TF = a(t^2 + 1) \qquad (A.16)$$

Referring fig.1, Δ TQP & Δ TOM are similar

$$\therefore \frac{PQ}{TQ} = \frac{OM}{OT}$$

$$\therefore OM = \frac{PQ \times OT}{TQ}$$

Substituting eqns. (A.2), (A.11) & (A.12) in above,

$$\therefore OM = \frac{2at \times at^2}{2at^2}$$

$$\therefore OM = at \quad (A.17)$$

Referring fig.1, in right-triangle FOM,

$$FM^2 = OF^2 + OM^2$$

Substituting eqns. (A.4) and (A.17) in above,

$$\therefore FM^2 = a^2 + (at)^2$$

$$\therefore FM^2 = a^2 + a^2t^2$$

$$\therefore FM^2 = a^2(t^2 + 1)$$

$$\therefore FM = a\sqrt{t^2 + 1} \quad (A.18)$$

Referring fig.1, in right-triangle FOM,

$$TF^2 = TM^2 + FM^2$$

$$\therefore TM^2 = TF^2 - FM^2$$

Substituting eqns. (A.16) and (A.18) in above,

$$\therefore TM^2 = [a(t^2 + 1)]^2 - \left(a\sqrt{t^2 + 1}\right)^2$$

$$\therefore TM^2 = a^2(t^2 + 1)^2 - a^2(t^2 + 1)$$

$$\therefore TM^2 = a^2(t^2 + 1)[t^2 + 1 - 1]$$

$$\therefore TM^2 = a^2(t^2 + 1)(t^2)$$

$$\therefore TM^2 = a^2t^2(t^2 + 1)$$

$$\therefore TM = at\sqrt{t^2 + 1} \quad (A.19)$$

Referring fig.1, in right-triangle PMF,

$$FP^2 = FM^2 + PM^2$$

$$\therefore PM^2 = FP^2 - FM^2$$

Substituting eqns. (A.9) and (A.18) in above,

$$\therefore PM^2 = [a(t^2+1)]^2 - \left(a\sqrt{t^2+1}\right)^2$$

$$\therefore PM^2 = a^2(t^2+1)^2 - a^2(t^2+1)$$

$$\therefore PM^2 = a^2(t^2+1)(t^2+1-1)$$

$$\therefore PM^2 = a^2(t^2+1)(t^2)$$

$$\therefore PM^2 = a^2t^2(t^2+1)$$

$$\therefore PM = at\sqrt{t^2+1} \qquad (A.20)$$

Referring fig.1, $TA = OT - OA$

Substituting eqns. (A.11) and (A.4) in above,

$$\therefore TA = at^2 - a$$

$$\therefore TA = a(t^2 - 1) \qquad (A.21)$$

Referring fig.1, in right-triangle QOM,

$$QM^2 = OQ^2 + OM^2$$

Substituting eqns. (A.1) and (A.17) in above,

$$\therefore QM^2 = (at^2)^2 + (at)^2$$

$$\therefore QM^2 = a^2t^4 + a^2t^2$$

$$\therefore QM^2 = a^2t^2(t^2+1)$$

$$\therefore QM = at\sqrt{t^2+1} \qquad (A.22)$$

Referring fig.1, $FS = FQ + SQ$

Substituting eqns. (A.7) and (A.6) in above,

$\therefore FS = a(t^2 - 1) + 2a$

$\therefore FS = a[t^2 - 1 + 2]$

$$\therefore FS = a(t^2 + 1) \qquad (A.23)$$

Referring fig.1, in right-triangle QFG,

$QG^2 = FQ^2 + FG^2$

Substituting eqns. (A.7) and (A.5) in above,

$\therefore QG^2 = [a(t^2 - 1)]^2 + (2a)^2$

$\therefore QG^2 = a^2(t^2 - 1)^2 + 4a^2$

$\therefore QG^2 = a^2[(t^2 - 1)^2 + 4]$

$$\therefore QG = a\sqrt{(t^2 - 1)^2 + 4} \qquad (A.24)$$

Referring fig.1, $PR = 2PQ$

Substituting eqns. (A.2)

$\therefore PR = 2 \times 2at$

$$\therefore PR = 4at \qquad (A.25)$$

Referring fig. 5, $SG^2 = SF^2 + FG^2$

Substituting eqns. (A.23) & (A.5) in above eqn.,

$SG^2 = [a(t^2 + 1)]^2 + [2a]^2$

$\therefore SG^2 = [a(t^2 + 1)]^2 + [2a]^2$

$\therefore SG^2 = a^2(t^2 + 1)^2 + 4a^2$

$\therefore SG^2 = a^2[(t^2 + 1)^2 + 4]$

$$\therefore SG = a\sqrt{(t^2 + 1)^2 + 4} \qquad (A.26)$$

Referring fig. 1, $AS = TS - TA$

Substituting eqns. (A.15) & (A.21) in above eqn.,

$$\therefore AS = 2a(t^2 + 1) - a(t^2 - 1)$$

$$\therefore AS = a(2t^2 + 2 - t^2 + 1)$$

$$\therefore AS = a(t^2 + 3) \qquad (A.27)$$

Referring fig. 1, $AM^2 = OA^2 + OM^2$

Substituting eqns. (A.23) & (A.5) in above eqn.,

$$\therefore AM^2 = a^2 + (at)^2$$

$$\therefore AM^2 = a^2 + a^2t^2$$

$$\therefore AM^2 = a^2(t^2 + 1)$$

$$\therefore AM = a\sqrt{t^2 + 1} \qquad (A.28)$$

Referring fig. 1, $AQ = AF + FQ$

Substituting eqns. (A.3) & (A.7) in above eqn.,

$$\therefore AQ = 2a + a(t^2 - 1)$$

$$\therefore AQ = a(2 + t^2 - 1)$$

$$\therefore AQ = a(t^2 + 1) \qquad (A.29)$$

PREAMBLE- B

Referring fig. 2, Δ TAN & Δ TQP are similar

$$\therefore \frac{AN}{TA} = \frac{PQ}{TQ}$$

$$\therefore AN = \frac{PQ \times TA}{TQ}$$

Substituting eqns. (A.2), (A.21) & (A.12) in above,

$$\therefore AN = \frac{2at \times a(t^2 - 1)}{2at^2}$$

$$\therefore AN = \frac{a(t^2 - 1)}{t} \qquad (B.1)$$

Referring fig. 2, in right-triangle FAN,

$$FN^2 = AF^2 + AN^2$$

Substituting eqns. (A.3) and (A.22) in above,

$$FN^2 = (2a)^2 + \left(\frac{a(t^2 - 1)}{t}\right)^2$$

$$\therefore FN^2 = 4a^2 + \left(\frac{a^2(t^2 - 1)^2}{t^2}\right)$$

$$\therefore FN^2 = \frac{4a^2t^2 + a^2(t^2 - 1)^2}{t^2}$$

$$\therefore FN^2 = \frac{4a^2t^2 + a^2(t^4 + 1 - 2t^2)}{t^2}$$

$$\therefore FN^2 = \frac{a^2(4t^2 + t^4 + 1 - 2t^2)}{t^2}$$

$$\therefore FN^2 = \frac{a^2(t^4 + 1 + 2t^2)}{t^2}$$

$$\therefore FN^2 = \frac{a^2(t^2 + 1)^2}{t^2}$$

$$\therefore FN = \frac{a(t^2 + 1)}{t} \qquad (B.2)$$

Referring fig. 2, in right-triangle TAN,

$$TN^2 = AT^2 + AN^2$$

Substituting eqns. (A.21) and (A.22) in above,

$$\therefore TN^2 = [a(t^2 - 1)]^2 + \left(\frac{a(t^2 - 1)}{t}\right)^2$$

$$\therefore TN^2 = a^2(t^2 - 1)^2 + \left(\frac{a(t^2 - 1)}{t}\right)^2$$

$$\therefore TN^2 = a^2(t^2 - 1)^2 + \frac{a^2(t^2 - 1)^2}{t^2}$$

$$\therefore TN^2 = \frac{a^2(t^2 - 1)^2t^2 + a^2(t^2 - 1)^2}{t^2}$$

$$\therefore TN^2 = \frac{a^2(t^2-1)^2(t^2+1)}{t^2}$$

$$\therefore TN = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \tag{B.3}$$

Referring fig. 2,

$$PN = PT - TN$$

Substituting eqns. (A.13) and (A.27) in above,

$$\therefore PN = 2at\sqrt{t^2+1} - \left(\frac{a(t^2-1)\sqrt{t^2+1}}{t}\right)$$

$$\therefore PN = \frac{2at^2\sqrt{t^2+1} - a(t^2-1)\sqrt{t^2+1}}{t}$$

$$\therefore PN = \frac{a\sqrt{t^2+1}[2t^2 - t^2 + 1]}{t}$$

$$\therefore PN = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \tag{B.4}$$

Referring fig. 2, Δ TPS & Δ TUQ are similar,

$$\therefore \frac{PS}{TS} = \frac{UQ}{TQ}$$

$$\therefore UQ = \frac{PS \times TQ}{TS}$$

Substituting eqns. (A.14), (A.12) and (A.15) in above,

$$\therefore UQ = \frac{2a\sqrt{t^2+1} \times 2at^2}{2a(t^2+1)}$$

$$\therefore UQ = \frac{2at^2}{\sqrt{t^2+1}} \tag{B.5}$$

Referring fig. 2, in right-triangle PUQ,

$$\therefore PU^2 = PQ^2 - UQ^2$$

Substituting eqns. (A.2) and (B.5) in above,

$$\therefore PU^2 = (2at)^2 - \left(\frac{2at^2}{\sqrt{t^2+1}}\right)^2$$

$$\therefore PU^2 = 4a^2t^2 - \left(\frac{4a^2t^4}{t^2+1}\right)$$

$$\therefore PU^2 = \left(\frac{4a^2t^2(t^2+1) - 4a^2t^4}{t^2+1}\right)$$

$$\therefore PU^2 = \frac{4a^2t^2(t^2+1-t^2)}{t^2+1}$$

$$\therefore PU^2 = \frac{4a^2t^2}{t^2+1}$$

$$\therefore PU = \frac{2at}{\sqrt{t^2+1}} \qquad (B.6)$$

Referring fig. 2, Δ TPS & Δ TVO are similar,

$$\therefore \frac{PS}{TS} = \frac{OV}{OT}$$

$$\therefore OV = \frac{PS \times OT}{TS}$$

Substituting eqns. (A.14), (A.11) and (A.15) in above,

$$\therefore OV = \frac{2a\sqrt{t^2+1} \times at^2}{2a(t^2+1)}$$

$$\therefore OV = \frac{at^2}{\sqrt{t^2+1}} \qquad (B.7)$$

Referring fig. 2, Δ PNJ & Δ PMF are similar,

$$\therefore \frac{JN}{PN} = \frac{FM}{PM}$$

$$\therefore JN = \frac{FM \times PN}{PM}$$

Substituting eqns. (A.18), (A.11) and (A.20) in above,

$$\therefore \mathrm{JN} = \frac{a\sqrt{t^2+1} \times \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t}\right)}{at\sqrt{t^2+1}}$$

$$\therefore \mathrm{JN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \times \frac{1}{t}$$

$$\therefore \mathrm{JN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \qquad (B.8)$$

Referring fig. 2, Δ TPS & Δ TLA are similar,

$$\therefore \frac{PS}{TS} = \frac{AL}{TA}$$

$$\therefore AL = \frac{PS \times TA}{TS}$$

Substituting eqns. (A.14), (A.21) and (A.15) in above,

$$\therefore AL = \frac{2a\sqrt{t^2+1} \times a(t^2-1)}{2a(t^2+1)}$$

$$\therefore AL = \frac{a(t^2-1)}{\sqrt{t^2+1}} \qquad (B.9)$$

Referring fig. 2, Δ PQT & Δ ZFT are similar

$$\therefore \frac{FZ}{TF} = \frac{PQ}{TQ}$$

$$\therefore FZ = \frac{PQ \times TF}{TQ}$$

Substituting eqns. (A.2), (A.16) & (A.12) in above,

$$\therefore FZ = \frac{2at \times a(t^2+1)}{2at^2}$$

$$\therefore FZ = \frac{a(t^2+1)}{t} \qquad (B.10)$$

Referring fig. 2, $NM = TM - TN$

Substituting eqns. (A.19) & (B.3) in above,

$$NM = at\sqrt{t^2+1} - \frac{a(t^2-1)\sqrt{t^2+1}}{t}$$

$$\therefore NM = \frac{at^2\sqrt{t^2+1} - a(t^2-1)\sqrt{t^2+1}}{t}$$

$$\therefore NM = \frac{a\sqrt{t^2+1}(t^2-t^2+1)}{t}$$

$$\therefore NM = \frac{a\sqrt{t^2+1}}{t} \qquad (B.11)$$

PREAMBLE- C

Referring fig. 3, Δ TPS & Δ OBS are similar

$$\frac{PT}{TS} = \frac{OB}{OS}$$

$$\therefore OB = \frac{PT \times OS}{TS}$$

Substituting eqns. (A.13), (A.10) & (A.15) in above,

$$\therefore OB = \frac{2at\sqrt{t^2+1} \times a(t^2+2)}{2a(t^2+1)}$$

$$\therefore OB = \frac{at \times (t^2+2)}{\sqrt{t^2+1}} \qquad (C.1)$$

Referring fig. 3, Δ TPS & Δ FCS are similar

$$\frac{PT}{TS} = \frac{FC}{FS}$$

$$\therefore FC = \frac{PT \times FS}{TS}$$

Substituting eqns. (A.13), (A.23) & (A.15) in above,

$$\therefore FC = \frac{2at\sqrt{t^2+1} \times a(t^2+1)}{2a(t^2+1)}$$

$$\therefore FC = at\sqrt{t^2+1} \qquad (C.2)$$

Referring fig. 3, Δ TPS & Δ FCS are similar

$$\frac{PT}{TS} = \frac{QD}{SQ}$$

$$\therefore QD = \frac{PT \times SQ}{TS}$$

Substituting eqns. (A.13), (A.23) & (A.15) in above,

$$\therefore QD = \frac{2at\sqrt{t^2 + 1} \times 2a}{2a(t^2 + 1)}$$

$$\therefore QD = \frac{2at}{\sqrt{t^2 + 1}} \quad (C.3)$$

In addition to above derivation, referring fig. 3, $QD = PU$

Substituting eqn. (A.26),

$$\therefore QD = \frac{2at}{\sqrt{t^2 + 1}} \quad (C.4)$$

Comparing eqns. (C.3) & (C.4), both are same results.

Referring fig. 3, Δ TPS & Δ FCS are similar

$$\frac{PQ}{QD} = \frac{PR}{RK}$$

$$\therefore RK = \frac{PR \times QD}{PQ}$$

Substituting eqns. (A.25), (C.3) & (A.2) in above,

$$\therefore RK = \frac{4at \times 2a}{2at}$$

$$\therefore RK = 4a \quad (C.5)$$

Referring fig. 3, in right-triangle PKR,

$$PR^2 = PK^2 + RK^2$$

$$\therefore PK^2 = PR^2 - RK^2$$

Substituting eqns. (A.25) & (C.5) in above,

$$\therefore PK^2 = (4at)^2 - (4a)^2$$

$$\therefore PK^2 = 16a^2t^2 - 16a^2$$

$$\therefore PK^2 = 16a^2(t^2 - 1)$$

$$\therefore PK = 4a\sqrt{t^2 - 1} \quad (C.6)$$

Referring fig. 3, SK = PK − PS

Substituting eqns. (C.6) & (A.14) in above,

$$\therefore SK = 4a\sqrt{t^2 - 1} - 2a\sqrt{t^2 + 1}$$

$$\therefore SK = 2a\left(2\sqrt{t^2 - 1} - \sqrt{t^2 + 1}\right) \quad (C.7)$$

Referring fig. 3, in right-triangle OBS,

$$OS^2 = SB^2 + OB^2$$

$$\therefore SB^2 = OS^2 - OB^2$$

Substituting eqns. (A.10) & (C.1) in above,

$$\therefore SB^2 = \left(a(t^2 + 2)\right)^2 - \left(\frac{at \times (t^2 + 2)}{\sqrt{t^2 + 1}}\right)^2$$

$$\therefore SB^2 = a^2(t^2 + 2)^2 - \left(\frac{a^2t^2 \times (t^2 + 2)^2}{t^2 + 1}\right)$$

$$\therefore SB^2 = \frac{a^2(t^2 + 1)(t^2 + 2)^2 - a^2t^2 \times (t^2 + 2)^2}{t^2 + 1}$$

$$\therefore SB^2 = \frac{a^2(t^2 + 2)^2[t^2 + 1 - t^2]}{t^2 + 1}$$

$$\therefore SB^2 = \frac{a^2(t^2 + 2)^2}{t^2 + 1}$$

$$\therefore SB = \frac{a(t^2 + 2)}{\sqrt{t^2 + 1}} \quad (C.8)$$

Referring fig. 3, in right-triangle OBS,

$$FS^2 = SC^2 + FC^2$$

$$\therefore SC^2 = FS^2 - FC^2$$

Substituting eqns. (A.23) & (C.2) in above,

$$\therefore SC^2 = [a(t^2+1)]^2 - \left(at\sqrt{t^2+1}\right)^2$$

$$\therefore SC^2 = a^2(t^2+1)^2 - a^2t^2(t^2+1)$$

$$\therefore SC^2 = a^2(t^2+1)[t^2+1-t^2]$$

$$\therefore SC^2 = a^2(t^2+1)$$

$$\therefore SC = a\sqrt{t^2+1} \qquad (C.9)$$

Referring fig. 3, Δ TPS & Δ AES are similar

$$\frac{PT}{TS} = \frac{AE}{AS}$$

$$\therefore AE = \frac{PT \times AS}{TS}$$

Substituting eqns. (A.13), (C.10) & (A.15) in above,

$$\therefore AE = \frac{2at\sqrt{t^2+1} \times a(t^2+3)}{2a(t^2+1)}$$

$$\therefore AE = \frac{at(t^2+3)}{\sqrt{t^2+1}} \qquad (C.10)$$

Referring fig. 3, PC = PS − SC

Substituting eqns. (A.14) & (C.9) in above,

$$\therefore PC = 2a\sqrt{t^2+1} - a\sqrt{t^2+1}$$

$$\therefore PC = a\sqrt{t^2+1}$$

Referring fig. 3, Δ TPS & Δ AES are similar

$$\frac{PS}{TS} = \frac{SE}{AS}$$

$$\therefore SE = \frac{PS \times AS}{TS}$$

Substituting eqns. (A.14), (C.10) & (A.15) in above,

$$\therefore SE = \frac{2a\sqrt{t^2+1} \times a(t^2+3)}{2a(t^2+1)}$$

$$\therefore SE = \frac{a(t^2+3)}{\sqrt{t^2+1}} \quad (C.11)$$

Referring fig. 3, PE = PS − SE

Substituting eqns. (A.14) & (C.11) in above,

$$\therefore PE = 2a\sqrt{t^2+1} - \left(\frac{a(t^2+3)}{\sqrt{t^2+1}}\right)$$

$$\therefore PE = \frac{2a(t^2+1) - a(t^2+3)}{\sqrt{t^2+1}}$$

$$\therefore PE = \frac{a(2t^2+2-t^2-3)}{\sqrt{t^2+1}}$$

$$\therefore PE = \frac{a(t^2-1)}{\sqrt{t^2+1}} \quad (C.12)$$

Referring fig. 3, PB = PS − SB

Substituting eqns. (A.14) & (C.8) in above,

$$\therefore PB = 2a\sqrt{t^2+1} - \frac{a(t^2+2)}{\sqrt{t^2+1}}$$

$$\therefore PB = \frac{2a(t^2+1) - a(t^2+2)}{\sqrt{t^2+1}}$$

$$\therefore PB = \frac{a(2t^2+2-t^2-2)}{\sqrt{t^2+1}}$$

$$\therefore PB = \frac{at^2}{\sqrt{t^2+1}} \quad (C.13)$$

PREAMBLE- D

Referring fig. 4, in right-triangle FQP, let ∠ PFQ = θ°

$$FP = \frac{2a}{1-\cos(\theta°)} \quad (D.1)$$

$$FJ = \frac{2a}{1-\cos(180+\theta°)}$$

$$\therefore FJ = \frac{2a}{1 + \cos(\theta°)} \tag{D.2}$$

Adding reciprocals of eqns. (D.1) & (D.2),

$$\frac{1}{|FP|} + \frac{1}{|FJ|} = \left(\frac{1 - \cos(\theta°)}{2a}\right) + \left(\frac{1 + \cos(\theta°)}{2a}\right)$$

$$\therefore \frac{1}{|FP|} + \frac{1}{|FJ|} = \frac{1 - \cos(\theta°) + 1 + \cos(\theta°)}{2a}$$

$$\therefore \frac{1}{|FP|} + \frac{1}{|FJ|} = \frac{2}{2a}$$

$$\therefore \frac{1}{|FP|} + \frac{1}{|FJ|} = \frac{1}{a} \tag{D.3}$$

Referring fig. 1,

$$GH = FG + FH$$

Substituting eqn. (A.5) in above,

$$\therefore GH = 4a$$

$$\therefore \frac{GH}{4} = a$$

Substituting the above eqn. in (D.3),

$$\frac{1}{FP} + \frac{1}{FJ} = \frac{1}{\left(\frac{GH}{4}\right)}$$

$$\frac{1}{FP} + \frac{1}{FJ} = 4 \times \frac{1}{|GH|} \tag{D.4}$$

Referring fig. 1,

$$\text{Eqn.}(D.3) \Longrightarrow \frac{1}{FP} + \frac{1}{FJ} = \frac{1}{a}$$

$$\text{Eqn.}(A.9) \Longrightarrow FP = a(t^2 + 1)$$

Substituting eqn. (A.9) in (D.3) in above,

$$\frac{1}{a(t^2+1)} + \frac{1}{FJ} = \frac{1}{a}$$

$$\therefore \frac{1}{FJ} = \frac{1}{a} - \left(\frac{1}{a(t^2+1)}\right)$$

$$\therefore \frac{1}{FJ} = \frac{t^2+1-1}{a(t^2+1)}$$

$$\therefore \frac{1}{FJ} = \frac{t^2}{a(t^2+1)}$$

$$\therefore FJ = \frac{a(t^2+1)}{t^2} \qquad (D.5)$$

Referring fig. 1,

$PJ = FP + FJ$

Substituting eqn. (A.9) in (D.5) in above,

$$\therefore PJ = a(t^2+1) + \left(\frac{a(t^2+1)}{t^2}\right)$$

$$\therefore PJ = \frac{at^2(t^2+1) + a(t^2+1)}{t^2}$$

$$\therefore PJ = \frac{a(t^2+1)(t^2+1)}{t^2}$$

$$\therefore PJ = \frac{a(t^2+1)^2}{t^2} \qquad (D.6)$$

Referring fig. 4. We know that $\angle PFQ = \theta°$.

In right-triangle QFP,

$$\sin(\theta°) = \frac{PQ}{FP}$$

Substituting eqns. (A.2) & (A.9) in above eqn.,

$$\sin(\theta°) = \frac{2at}{a(t^2+1)}$$

$$\therefore \sin(\theta°) = \frac{2t}{t^2+1} \tag{D.7}$$

Referring fig. 5, if $\angle PTQ = \theta°$, $\angle JFW$ is also equal to $\theta°$

In right-triangle JWF,

$$\sin(\theta°) = \frac{JW}{FJ}$$

$$\therefore JW = FJ \times \sin(\theta°)$$

Substituting eqns. (D.5) & (D.7) in above,

$$\therefore JW = \left(\frac{a(t^2+1)}{t^2}\right) \times \left(\frac{2t}{t^2+1}\right)$$

$$\therefore JW = \frac{2a}{t} \tag{D.8}$$

Referring fig. 4. We know that $\angle PFQ = \theta°$.

In right-triangle QFP,

$$\cos(\theta°) = \frac{FQ}{FP}$$

Substituting eqns. (A.7) & (A.9) in above eqn.,

$$\cos(\theta°) = \frac{a(t^2-1)}{a(t^2+1)}$$

$$\therefore \cos(\theta°) = \frac{t^2-1}{t^2+1} \tag{D.9}$$

Referring fig. 6, if $\angle PTQ = \theta°$, $\angle JFW$ is also equal to $\theta°$

In right-triangle JWF,

$$\cos(\theta°) = \frac{FW}{FJ}$$

$$\therefore FW = FJ \times \cos(\theta°)$$

Substituting eqns. (D.5) & (D.9) in above,

$$\therefore \text{FW} = \left(\frac{a(t^2+1)}{t^2}\right) \times \left(\frac{t^2-1}{t^2+1}\right)$$

$$\therefore \text{FW} = \frac{a(t^2-1)}{t^2} \tag{D.10}$$

Referring fig. 6, $\text{OW} = \text{OF} - \text{FW}$

Substituting eqns. (A.4) and (D.10) in above,

$$\therefore \text{OW} = a - \left(\frac{a(t^2-1)}{t^2}\right)$$

$$\therefore \text{OW} = \frac{at^2 - a(t^2-1)}{t^2}$$

$$\therefore \text{OW} = \frac{at^2 - at^2 + 1}{t^2}$$

$$\therefore \text{OW} = \frac{1}{t^2} \tag{D.11}$$

In right-triangle FQP

$$\tan(\theta°) = \frac{\text{PQ}}{\text{FQ}}$$

Substituting eqns. (A.2) and (A.7) in above,

$$\therefore \tan(\theta°) = \frac{2at}{a(t^2-1)}$$

$$\therefore \tan(\theta°) = \frac{2t}{t^2-1} \tag{D.12}$$

Referring fig. 5, $\text{TW} = \text{FT} - \text{FW}$

Substituting eqns. (A.16) and (D.10) in above,

$$\text{TW} = a(t^2+1) - \frac{a(t^2-1)}{t^2}$$

$$\therefore \text{TW} = \frac{at^2(t^2+1) - a(t^2-1)}{t^2}$$

$$\therefore \text{TW} = \frac{a(t^4 + t^2 - t^2 + 1)}{t^2}$$

$$\therefore TW = \frac{a(t^4+1)}{t^2} \quad (D.13)$$

General formula to calculate $\tan\left(\frac{\theta°}{2}\right)$ is $\frac{\sin(\theta°)}{1+\cos(\theta°)}$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{\sin(\theta°)}{1+\cos(\theta°)}$$

Substituting eqns. (D.7) & (D.9) in above eqn.,

$$\tan\left(\frac{\theta°}{2}\right) = \frac{\frac{2t}{t^2+1}}{1+\left(\frac{t^2-1}{t^2+1}\right)}$$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{\frac{2t}{t^2+1}}{\frac{t^2+1+t^2-1}{t^2+1}}$$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{2t}{t^2+1} \times \frac{t^2+1}{2t^2}$$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{2t}{t^2+1} \times \frac{t^2+1}{2t^2}$$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{1}{t} \quad (D.14)$$

Referring fig. 3, in right triangle TWI, ∠ WTI is equal to $\frac{\theta°}{2}$

$$\tan\left(\frac{\theta°}{2}\right) = \frac{WI}{TW}$$

$$\therefore WI = TW \times \tan\left(\frac{\theta°}{2}\right)$$

Substituting eqns. (D.13) & (D.14) in above eqn.,

$$\therefore WI = \frac{a(t^4+1)}{t^2} \times \frac{1}{t}$$

$$\therefore WI = \frac{a(t^4+1)}{t^3} \quad (D.15)$$

General formula to calculate $\sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{1-\cos(\theta°)}{2}}$

Substituting eqn. (D.9) in above eqn.,

$$\sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{1-\left(\frac{t^2-1}{t^2+1}\right)}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{\frac{t^2+1-(t^2-1)}{t^2+1}}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{\frac{t^2+1-t^2+1}{t^2+1}}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{\frac{t^2+1-t^2+1}{t^2+1}}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{\frac{2}{t^2+1}}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{2}{t^2+1} \times \frac{1}{2}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \sqrt{\frac{1}{t^2+1}}$$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \frac{1}{\sqrt{t^2+1}} \qquad (D.16)$$

Referring fig. 5. In right-triangle SWZ,

$SZ^2 = FS^2 + FZ^2$

Substituting eqns. (A.23) & (B.10),

$$\therefore SZ^2 = [a(t^2+1)]^2 + \left[\frac{a(t^2+1)}{t}\right]^2$$

$$\therefore SZ^2 = a^2(t^2+1)^2 + \frac{a^2(t^2+1)^2}{t^2}$$

$$\therefore SZ^2 = \frac{a^2t^2(t^2+1)^2 + a^2(t^2+1)^2}{t^2}$$

$$\therefore SZ^2 = \frac{a^2(t^2+1)^2\,(t^2+1)}{t^2}$$

$$\therefore SZ = \frac{a\,(t^2+1)\sqrt{t^2+1}}{t} \qquad (D.17)$$

General formula to calculate $\cos\left(\frac{\theta°}{2}\right)$ is $\sqrt{\frac{1+\cos(\theta°)}{2}}$

Substituting eqn. (D.9) in above eqn.,

$$\cos\left(\frac{\theta°}{2}\right) = \sqrt{\frac{1+\left(\frac{t^2-1}{t^2+1}\right)}{2}}$$

$$\therefore \cos\left(\frac{\theta°}{2}\right) = \sqrt{\frac{\frac{t^2+1+(t^2-1)}{t^2+1}}{2}}$$

$$\therefore \cos\left(\frac{\theta°}{2}\right) = \sqrt{\frac{2t^2}{t^2+1}\times\frac{1}{2}}$$

$$\therefore \cos\left(\frac{\theta°}{2}\right) = \sqrt{\frac{t^2}{t^2+1}} \qquad (D.18)$$

Referring fig. 5, in right triangle OVM, $\angle VOM = \frac{\theta°}{2}$

$$\sin\left(\frac{\theta°}{2}\right) = \frac{VM}{OM}$$

$$\therefore VM = OM \times \sin\left(\frac{\theta°}{2}\right)$$

Substituting eqns. (A.17) & (D.17) in above eqn.,

$$\therefore \text{VM} = \text{at} \times \frac{1}{\sqrt{t^2+1}}$$

$$\therefore \text{VM} = \frac{\text{at}}{\sqrt{t^2+1}} \qquad (D.19)$$

Referring fig. 3, $\angle \text{QPS} = \frac{\theta°}{2}$

$$\therefore \cos\left(\frac{\theta°}{2}\right) = \frac{\text{PD}}{\text{PQ}}$$

$$\therefore \text{PD} = \text{PQ} \times \cos\left(\frac{\theta°}{2}\right)$$

Substituting eqns. (A.2) & (D.18) in above,

$$\therefore \text{PD} = 2\text{at} \times \frac{t}{\sqrt{t^2+1}}$$

$$\therefore \text{PD} = \frac{2\text{at}^2}{\sqrt{t^2+1}} \qquad (D.20)$$

Referring fig. 3, SD = PS − PD

Substituting eqns. (A.14) & (D.24) in above,

$$\therefore \text{SD} = 2a\sqrt{t^2+1} - \frac{2\text{at}^2}{\sqrt{t^2+1}}$$

$$\therefore \text{SD} = \frac{2a(t^2+1) - 2\text{at}^2}{\sqrt{t^2+1}}$$

$$\therefore \text{SD} = \frac{2a(t^2+1-t^2)}{\sqrt{t^2+1}}$$

$$\therefore \text{SD} = \frac{2a}{\sqrt{t^2+1}} \qquad (D.21)$$

Referring fig. 5, AW = TW − TA

Substituting eqns. (D.13) & (A.21) in above,

$$\therefore \text{AW} = \frac{a(t^4+1)}{t^2} - a(t^2-1)$$

$$\therefore AW = \frac{a(t^4+1) - at^2(t^2-1)}{t^2}$$

$$\therefore AW = \frac{at^4 + a - at^4 + at^2}{t^2}$$

$$\therefore AW = \frac{at^2 + a}{t^2}$$

$$\therefore AW = \frac{a(t^2+1)}{t^2} \quad (D.22)$$

Referring fig. 5, QW = TQ − TW

Substituting eqns. (A.12) & (D.13) in above,

$$\therefore QW = 2at^2 - \frac{a(t^4+1)}{t^2}$$

$$\therefore QW = \frac{2at^4 - a(t^4+1)}{t^2}$$

$$\therefore QW = \frac{a(t^4-1)}{t^2} \quad (D.23)$$

PREAMBLE- E

Referring fig. 5,

In right triangle FQP, $\angle QFP = \theta°$

If $\angle QFP = \theta°$, $\quad \angle QFX = 90 - \frac{\theta°}{2}$

We already know that as per eqn. (D.1), $FP = \frac{2a}{1 - \cos(\theta°)}$

$$\therefore FX = \frac{2a}{1 - \cos\left(90 - \frac{\theta°}{2}\right)}$$

$$\therefore FX = \frac{2a}{1 - \sin\left(\frac{\theta°}{2}\right)} \quad (E.1)$$

Eqn. (D.9) $\Rightarrow \cos(\theta°) = \frac{t^2-1}{t^2+1}$

Substituting above eqn. in (E.1),

$$FX = \frac{2a}{\left(1 - \frac{1}{\sqrt{t^2 + 1}}\right)}$$

$$\therefore FX = \frac{2a}{\left(\frac{\sqrt{t^2 + 1} - 1}{\sqrt{t^2 + 1}}\right)}$$

$$\therefore FX = \frac{2a\sqrt{t^2 + 1}}{\sqrt{t^2 + 1} - 1}$$

$$\therefore FX = \frac{2a\sqrt{t^2 + 1}}{\sqrt{t^2 + 1} - 1} \qquad (E.2)$$

In right triangle FQP, $\angle QFP = \theta°$

If $\angle QFP = \theta°$, $\quad \angle QFX = \left(90 + \frac{\theta°}{2}\right) + 180 = 270 + \frac{\theta°}{2}$

We already know that as per eqn. (D. 1), $FP = \frac{2a}{1 - \cos(\theta°)}$

$$\therefore FY = \frac{2a}{\left[1 + \cos\left(270 + \frac{\theta°}{2}\right)\right]}$$

$$\therefore FY = \frac{2a}{\left[1 + \sin\left(\frac{\theta°}{2}\right)\right]}$$

Substituting eqn. (D.17) in above eqn.,

$$FY = \frac{2a}{\left(1 + \frac{1}{\sqrt{t^2 + 1}}\right)}$$

$$\therefore FY = \frac{2a}{\left(\frac{\sqrt{t^2 + 1} + 1}{\sqrt{t^2 + 1}}\right)}$$

$$\therefore FY = \frac{2a\sqrt{t^2 + 1}}{\sqrt{t^2 + 1} + 1} \qquad (E.3)$$

Referring fig. 5, $XY = FX + FY$

Substituting eqns. (D.18) & (D.19) in above eqn.,

$$\therefore XY = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}-1} + \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}+1}$$

$$\therefore XY = \frac{2a\sqrt{t^2+1} \times \left(\sqrt{t^2+1}+1\right) + 2a\sqrt{t^2+1} \times \left(\sqrt{t^2+1}-1\right)}{\left(\sqrt{t^2+1}-1\right) \times \left(\sqrt{t^2+1}+1\right)}$$

$$\therefore XY = \frac{2a\sqrt{t^2+1} \times \sqrt{t^2+1} + 2a\sqrt{t^2+1} + 2a\sqrt{t^2+1} \times \sqrt{t^2+1} - 2a\sqrt{t^2+1}}{t^2+1-1}$$

$$\therefore XY = \frac{4a\sqrt{t^2+1}\sqrt{t^2+1}}{t^2}$$

$$\therefore XY = \frac{4a(t^2+1)}{t^2} \qquad (E.4)$$

Referring fig. 2, Δ OVM & Δ μNM are equal

$$\therefore \frac{VM}{OM} = \frac{NM}{M\mu}$$

$$\therefore M\mu = \frac{NM \times OM}{VM}$$

Substituting eqns. (B.11), (A.17) & (D.23) in above eqn.,

$$\therefore M\mu = \left(\frac{a\sqrt{t^2+1}}{t} \times at\right) \div \left(\frac{at}{\sqrt{t^2+1}}\right)$$

$$\therefore M\mu = \left(\frac{a\sqrt{t^2+1}}{t} \times at\right) \times \left(\frac{\sqrt{t^2+1}}{at}\right)$$

$$\therefore M\mu = \frac{a(t^2+1)}{t} \qquad (E.5)$$

Referring fig. 6, $O\mu = M\mu - OM$

Substituting eqns. (D.24) & (A.17) in above eqn.,

$$\therefore O\mu = \frac{a(t^2+1)}{t} - at$$

$$\therefore O\mu = \frac{a(t^2+1) - at^2}{t}$$

$$\therefore O\mu = \frac{at^2 + a - at^2}{t}$$

$$\therefore O\mu = \frac{a}{t} \quad (E.6)$$

Referring fig. 2, Δ OVM & Δ μNM are similar

$$\therefore \frac{OV}{OM} = \frac{N\mu}{M\mu}$$

$$\therefore N\mu = \frac{OV \times M\mu}{OM}$$

Substituting eqns. (B.7), (D.24) & (A.17) in above eqn.,

$$\therefore N\mu = \frac{\left(\frac{at^2}{\sqrt{t^2+1}}\right) \times \left(\frac{a(t^2+1)}{t}\right)}{at}$$

$$\therefore N\mu = \frac{a^2t(t^2+1)}{\sqrt{t^2+1}} \times \frac{1}{at}$$

$$\therefore N\mu = a\sqrt{t^2+1} \quad (E.7)$$

Referring fig. 6, $J\mu = JN - N\mu$

$$\therefore J\mu = JN - N\mu$$

Substituting eqns. (B.8) & (D.26) in above eqn.,

$$\therefore J\mu = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} - a\sqrt{t^2+1}$$

$$\therefore J\mu = \frac{a(t^2+1)\sqrt{t^2+1} - at^2\sqrt{t^2+1}}{t^2}$$

$$\therefore J\mu = \frac{a\sqrt{t^2+1}(t^2+1-t^2)}{t^2}$$

$$\therefore J\mu = \frac{a\sqrt{t^2+1}}{t^2} \quad (E.8)$$

Referring fig. 3, $\angle PJ\Omega = \frac{\theta°}{2}$, In right-triangle JΩP,

$$\cos\left(\frac{\theta^\circ}{2}\right) = \frac{J\Omega}{PJ}$$

$$\therefore J\Omega = PJ \times \cos\left(\frac{\theta^\circ}{2}\right)$$

Substituting eqns. (D.6) & (D.18) in above eqn.,

$$\therefore J\Omega = \frac{a(t^2+1)^2}{t^2} \times \frac{t}{\sqrt{t^2+1}}$$

$$\therefore J\Omega = \frac{a(t^2+1)^2}{t\sqrt{t^2+1}}$$

$$\therefore J\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \qquad (E.9)$$

Referring fig. 3, $\angle PJ\Omega = \frac{\theta^\circ}{2}$,

In right-triangle JΩP,

$$\sin\left(\frac{\theta^\circ}{2}\right) = \frac{P\Omega}{PJ}$$

$$\therefore P\Omega = PJ \times \sin\left(\frac{\theta^\circ}{2}\right)$$

Substituting eqns. (D.6) & (D.16) in above eqn.,

$$\therefore P\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \qquad (E.10)$$

Referring fig. 3, $S\Omega = PS - P\Omega$

Substituting eqns. (A.14) & (E.10) in above eqn.,

$$\therefore S\Omega = 2a\sqrt{t^2+1} - \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t^2}\right)$$

$$\therefore S\Omega = \frac{a\sqrt{t^2+1}(2t^2 - t^2 - 1)}{t^2}$$

$$\therefore S\Omega = \frac{a(t^2-1)\sqrt{t^2+1}}{t^2} \qquad (E.11)$$

Referring fig. 5, $\angle NT\lambda = \frac{\theta°}{2}$,

In right-triangle TNλ,

$$\cos\left(\frac{\theta°}{2}\right) = \frac{TN}{T\lambda}$$

$$\therefore T\lambda = \frac{TN}{\cos\left(\frac{\theta°}{2}\right)}$$

Substituting eqns. (B.3) & (D.18) in above eqn.,

$$\therefore T\lambda = \left(\frac{a(t^2-1)\sqrt{t^2+1}}{t}\right) \div \left(\frac{t}{\sqrt{t^2+1}}\right)$$

$$\therefore T\lambda = \left(\frac{a(t^2-1)\sqrt{t^2+1}}{t}\right) \times \left(\frac{\sqrt{t^2+1}}{t}\right)$$

$$\therefore T\lambda = \frac{a(t^2-1)(t^2+1)}{t^2}$$

$$\therefore T\lambda = \frac{a(t^4-1)}{t^2} \qquad (E.12)$$

Referring fig. 5, $A\lambda = T\lambda - TA$

Substituting eqns. (E.12) & (A.21) in above eqn.,

$$\therefore A\lambda = \left(\frac{a(t^4-1)}{t^2}\right) - a(t^2-1)$$

$$\therefore A\lambda = \frac{a(t^4-1) - at^2(t^2-1)}{t^2}$$

$$\therefore A\lambda = \frac{a(t^4-1) - a(t^4-t^2)}{t^2}$$

$$\therefore A\lambda = \frac{a(t^4-1-t^4+t^2)}{t^2}$$

$$\therefore A\lambda = \frac{a(t^2 - 1)}{t^2} \quad (E.13)$$

Referring fig. 5, $O\lambda = OT - T\lambda$

Substituting eqns. (A.11) & (E.12) in above eqn.,

$$\therefore O\lambda = at^2 - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore O\lambda = \frac{at^4 - a(t^4 - 1)}{t^2}$$

$$\therefore O\lambda = \frac{at^4 - at^4 + a}{t^2}$$

$$\therefore O\lambda = \frac{a}{t^2} \quad (E.14)$$

Referring fig. 6, $W\lambda = TW - T\lambda$

Substituting eqns. (A.11) & (D.13) in above eqn.,

$$\therefore W\lambda = \left(\frac{a(t^4 + 1)}{t^2}\right) - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore W\lambda = \frac{a(t^4 + 1) - a(t^4 - 1)}{t^2}$$

$$\therefore W\lambda = \frac{a(t^4 + 1 - t^4 + 1)}{t^2}$$

$$\therefore W\lambda = \frac{2a}{t^2} \quad (E.15)$$

Referring fig. 6, $F\lambda = FT - T\lambda$

Substituting eqns. (A.16) & (D.13) in above eqn.,

$$\therefore F\lambda = a(t^2 + 1) - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore F\lambda = \frac{at^2(t^2 + 1) - a(t^4 - 1)}{t^2}$$

$$\therefore F\lambda = \frac{a(t^4 + t^2 - t^4 + 1)}{t^2}$$

$$\therefore F\lambda = \frac{a(t^2 + 1)}{t^2} \tag{E.16}$$

Referring fig. 5, $Q\lambda = TQ - T\lambda$

Substituting eqns. (A.12) & (D.13) in above eqn.,

$$\therefore Q\lambda = 2at^2 - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore Q\lambda = \frac{2at^4 - a(t^4 - 1)}{t^2}$$

$$\therefore Q\lambda = \frac{a(t^4 + 1)}{t^2} \tag{E.17}$$

Referring fig. 5, $S\lambda = TS - T\lambda$

Substituting eqns. (A.15) & (D.13) in above eqn.,

$$\therefore S\lambda = 2a(t^2 + 1) - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore S\lambda = \frac{2at^2(t^2 + 1) - a(t^4 - 1)}{t^2}$$

$$\therefore S\lambda = \frac{a\left(2t^2(t^2 + 1) - (t^4 - 1)\right)}{t^2}$$

$$\therefore S\lambda = \frac{a(2t^4 + 2t^2 - t^4 + 1)}{t^2}$$

$$\therefore S\lambda = \frac{a(t^4 + 2t^2 + 1)}{t^2}$$

$$\therefore S\lambda = \frac{a(t^2 + 1)^2}{t^2} \tag{E.18}$$

Referring fig. 6, $J\lambda^2 = W\lambda^2 + JW^2$

Substituting eqns. (E.14) & (D.8) in above eqn.,

$$\therefore J\lambda^2 = \left(\frac{2a}{t^2}\right)^2 + \left(\frac{2a}{t}\right)^2$$

$$\therefore J\lambda^2 = \frac{4a^2}{t^4} + \frac{4a^2}{t^2}$$

$$\therefore J\lambda^2 = \frac{4a^2 + 4a^2t^2}{t^4}$$

$$\therefore J\lambda^2 = \frac{4a^2(t^2+1)}{t^4}$$

$$\therefore J\lambda = \frac{2a\sqrt{t^2+1}}{t^2} \quad (E.19)$$

Referring fig. 6, $\mu\lambda^2 = O\mu^2 + O\lambda^2$

Substituting eqns. (E.6) & (E.14) in above eqn.,

$$\therefore \mu\lambda^2 = \left(\frac{a}{t}\right)^2 + \left(\frac{a}{t^2}\right)^2$$

$$\therefore \mu\lambda^2 = \frac{a^2}{t^2} + \frac{a^2}{t^4}$$

$$\therefore \mu\lambda^2 = \frac{a^2t^2 + a^2}{t^4}$$

$$\therefore \mu\lambda^2 = \frac{a^2(t^2+1)}{t^4}$$

$$\therefore \mu\lambda = \frac{a\sqrt{t^2+1}}{t^2} \quad (E.20)$$

Referring fig. 7, $P\sigma = OQ$

In right triangle $M\sigma P, \angle MP\sigma = \frac{\theta°}{2}$

$$\therefore \tan\left(\frac{\theta°}{2}\right) = \frac{M\sigma}{P\sigma}$$

$$\therefore M\sigma = P\sigma \times \tan\left(\frac{\theta°}{2}\right)$$

Substituting eqns. (D.14) & $P\sigma = OQ$ in above eqn.,

$$\therefore M\sigma = at^2 \times \frac{1}{t}$$

$$\therefore M\sigma = at \tag{E.21}$$

Referring fig. 7,

In right triangle $P\sigma\partial, \angle \sigma\partial P = \frac{\theta°}{2}$

$$\tan\left(\frac{\theta°}{2}\right) = \frac{P\sigma}{\partial\sigma}$$

$$\therefore \partial\sigma = \frac{P\sigma}{\tan\left(\frac{\theta°}{2}\right)}$$

Substituting eqns. (D.14) & $P\sigma = 0Q = at^2$ in above eqn.,

$$\therefore \partial\sigma = \frac{0Q}{\left(\frac{1}{t}\right)}$$

$$\therefore \partial\sigma = at^2 \times t$$

$$\therefore \partial\sigma = at^3 \tag{E.22}$$

In right triangle $\tau\omega P, \angle \omega\tau P = \frac{\theta°}{2}$

$$\tan\left(\frac{\theta°}{2}\right) = \frac{\omega P}{\tau\omega}$$

$$\therefore \tau\omega = \frac{\omega P}{\tan\left(\frac{\theta°}{2}\right)}$$

Substituting eqns. (D.14) & $\omega P = FQ$

Substituting eqn. (A.7) in above, $\omega P = a(t^2 - 1)$

$$\therefore \tau\omega = \frac{a(t^2 - 1)}{\left(\frac{1}{t}\right)}$$

$$\therefore \tau\omega = at(t^2 - 1) \tag{E.23}$$

Referring fig. 7, $F\omega = PQ$

$$\therefore F\omega = 2at \tag{E.24}$$

Referring fig. 2, In right triangle $\angle TLA = 90° \ \& \ \angle ATL = \delta° = \frac{\theta°}{2}$

$$\therefore \sin\left(\frac{\theta°}{2}\right) = \frac{AL}{TA}$$

$$\therefore AL = TA \times \sin\left(\frac{\theta°}{2}\right)$$

Substituting eqns. (A.21) & (D.16) in above eqn.,

$$\therefore AL = a(t^2 - 1) \times \frac{1}{\sqrt{t^2+1}}$$

$$\therefore AL = \frac{a(t^2 - 1)}{\sqrt{t^2+1}} \tag{E.25}$$

Referring fig. 2, In right triangle $\angle TLA = 90° \ \& \ \angle ATL = \frac{\theta°}{2}$

$$\therefore \cos\left(\frac{\theta°}{2}\right) = \frac{TA}{TL}$$

$$\therefore TL = \frac{TA}{\cos\left(\frac{\theta°}{2}\right)}$$

Substituting eqns. (A.21) & (D.18) in above eqn.,

$$\therefore TL = a(t^2 - 1) \div \left(\frac{t}{\sqrt{t^2+1}}\right)$$

$$\therefore TL = a(t^2 - 1) \times \left(\frac{\sqrt{t^2+1}}{t}\right)$$

$$\therefore TL = \frac{a(t^2 - 1)\sqrt{t^2+1}}{t} \tag{E.26}$$

Referring fig. 2, $PL = PT - TL$

Substituting eqns. (A.13) & (E.26) in above eqn.,

$$\therefore PL = 2at\sqrt{t^2+1} - \left(\frac{a(t^2 - 1)\sqrt{t^2+1}}{t}\right)$$

$$\therefore \mathrm{PL} = \frac{2at^2\sqrt{t^2+1} - a(t^2-1)\sqrt{t^2+1}}{t}$$

$$\therefore \mathrm{PL} = \frac{a\sqrt{t^2+1}(2t^2 - t^2 + 1)}{t}$$

$$\therefore \mathrm{PL} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \qquad (E.27)$$

The following table is abstract of derived equations of various elements of a parabola

A.1	$OQ = at^2$	D.1	$FP = \frac{2a}{1-\cos(\theta°)}$
A.2	$OR = PQ = 2at$	D.2	$FJ = \frac{2a}{1+\cos(\theta°)}$
A.3	$AF = 2a$	D.3	$\frac{1}{FP} + \frac{1}{FJ} = \frac{1}{a}$
A.4	$OF = OA = \frac{AF}{2} = a$	D.5	$FJ = \frac{a(t^2+1)}{t^2}$
A.5	$FG = FH = 2a$	D.6	$PJ = \frac{a(t^2+1)^2}{t^2}$
A.6	$SQ = 2a$	D.7	$\sin(\theta°) = \frac{2t}{t^2+1}$
A.7	$FQ = a(t^2-1)$	D.8	$JW = \frac{2a}{t}$
A.8	$OP = at\sqrt{t^2+4}$	D.9	$\cos(\theta°) = \frac{t^2-1}{t^2+1}$
A.9	$FP = a(t^2+1)$	D.10	$FW = \frac{a(t^2-1)}{t^2}$
A.10	$OS = a(t^2+2)$	D.11	$OW = \frac{1}{t^2}$
A.11	$OT = at^2$	D.12	$\tan(\theta°) = \frac{2t}{t^2-1}$
A.12	$TQ = 2at^2$	D.13	$TW = \frac{a(t^4+1)}{t^2}$

A.13	$PT = 2at\sqrt{t^2+1}$	D.14	$\tan\left(\frac{\theta°}{2}\right) = \frac{1}{t}$
A.14	$PS = 2a\sqrt{t^2+1}$	D.15	$WI = \frac{a(t^4+1)}{t^3}$
A.15	$TS = 2a(t^2+1)$	D.16	$\sin\left(\frac{\theta°}{2}\right) = \frac{1}{\sqrt{t^2+1}}$
A.16	$FT = a(t^2+1)$	D.17	$SZ = \frac{a(t^2+1)\sqrt{t^2+1}}{t}$
A.17	$OM = at$	D.18	$\cos\left(\frac{\theta°}{2}\right) = \frac{t}{\sqrt{t^2+1}}$
A.18	$FM = a\sqrt{t^2+1}$	D.19	$VM = \frac{at}{\sqrt{t^2+1}}$
A.19	$TM = at\sqrt{t^2+1}$	D.20	$PD = \frac{2at^2}{\sqrt{t^2+1}}$
A.20	$PM = at\sqrt{t^2+1}$	D.21	$SD = \frac{2a}{\sqrt{t^2+1}}$
A.21	$TA = a(t^2-1)$	D.22	$AW = \frac{a(t^2+1)}{t^2}$
A.22	$QM = at\sqrt{t^2+1}$	D.23	$QW = \frac{a(t^4-1)}{t^2}$
A.23	$FS = a(t^2+1)$	E.2	$FX = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}-1}$
A.24	$QG = a\sqrt{(t^2-1)^2+4}$	E.3	$FY = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}+1}$
A.25	$PR = 4at$	E.4	$XY = \frac{4a(t^2+1)}{t^2}$
A.26	$SG = a\sqrt{(t^2+1)^2+4}$	E.5	$M\mu = \frac{a(t^2+1)}{t}$
A.27	$AS = a(t^2+3)$	E.6	$O\mu = \frac{a}{t}$
A.28	$AM = a\sqrt{t^2+1}$	E.7	$N\mu = a\sqrt{t^2+1}$

A.29	$AQ = a(t^2+1)$	E.8	$J\mu = \frac{a\sqrt{t^2+1}}{t^2}$
B.1	$AN = \frac{a(t^2-1)}{t}$	E.9	$J\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t}$
B.2	$FN = \frac{a(t^2+1)}{t}$	E.10	$P\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2}$
B.3	$TN = \frac{a(t^2-1)\sqrt{t^2+1}}{t}$	E.11	$S\Omega = \frac{a(t^2-1)\sqrt{t^2+1}}{t^2}$
B.4	$PN = \frac{a(t^2+1)\sqrt{t^2+1}}{t}$	E.12	$T\lambda = \frac{a(t^4-1)}{t^2}$
B.5	$UQ = \frac{2at^2}{\sqrt{t^2+1}}$	E.13	$A\lambda = \frac{a(t^2-1)}{t^2}$
B.6	$PU = \frac{2at}{\sqrt{t^2+1}}$	E.14	$O\lambda = \frac{a}{t^2}$
B.7	$OV = \frac{at^2}{\sqrt{t^2+1}}$	E.15	$W\lambda = \frac{2a}{t^2}$
B.8	$JN = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2}$	E.16	$F\lambda = \frac{a(t^2+1)}{t^2}$
B.9	$AL = \frac{a(t^2-1)}{\sqrt{t^2+1}}$	E.17	$Q\lambda = \frac{a(t^4+1)}{t^2}$
B.10	$FZ = \frac{a(t^2+1)}{t}$	E.18	$S\lambda = \frac{a(t^2+1)^2}{t^2}$
B.11	$NM = \frac{a\sqrt{t^2+1}}{t}$	E.19	$J\lambda = \frac{2a\sqrt{t^2+1}}{t^2}$
C.1	$OB = \frac{at(t^2+2)}{\sqrt{t^2+1}}$	E.20	$\mu\lambda = \frac{a\sqrt{t^2+1}}{t^2}$
C.2	$FC = at\sqrt{t^2+1}$	E.21	$M\sigma = at$
C.3	$QD = \frac{2at}{\sqrt{t^2+1}}$	E.22	$\partial\sigma = at^3$
C.5	$RK = 4a$	E.23	$\tau\omega = at(t^2-1)$

C.6	$PK = 4a\sqrt{t^2 - 1}$	E.24	$F\omega = 2at$
C.7	$SK = 2a\left(2\sqrt{t^2 - 1} - \sqrt{t^2 + 1}\right)$	E.25	$AL = \frac{a(t^2 - 1)}{\sqrt{t^2 + 1}}$
C.8	$SB = \frac{a(t^2 + 2)}{\sqrt{t^2 + 1}}$	E.26	$TL = \frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t}$
C.9	$SC = a\sqrt{t^2 + 1}$	E.27	$PL = \frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}$
C.10	$AE = \frac{at(t^2 + 3)}{\sqrt{t^2 + 1}}$	C.12	$PE = \frac{a(t^2 - 1)}{\sqrt{t^2 + 1}}$
C.11	$SE = \frac{a(t^2 + 3)}{\sqrt{t^2 + 1}}$	C.13	$PB = \frac{at^2}{\sqrt{t^2 + 1}}$

ANALYSIS & DERIVATIONS OF THE THEOREMS

THEOREM- 1:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$, then the t is called parameter of the parabola and its value is given with respect to $\boldsymbol{\varphi}°$ & $\boldsymbol{\delta}°$

$$\boldsymbol{t = 2cot(\varphi°) = cot(\delta°) = cot\left(\frac{\theta°}{2}\right)}$$

Derivations for proof of the theorem

Referring fig. 4, in right-triangle OQP, let, $\angle POQ = \varphi°$

$$\tan(\varphi°) = \frac{PQ}{OQ}$$

Substituting eqns. (A.2) and (A.1) in above,

$$\therefore \tan(\varphi°) = \frac{2at}{at^2}$$

$$\therefore \tan(\varphi°) = \frac{2}{t}$$

$$\therefore t = \frac{2}{\tan(\varphi°)} \quad (1.1)$$

Referring fig. 4, in right-triangle TQP, let, $\angle$ QTP $= \delta°$

$$\tan(\delta°) = \frac{PQ}{TQ}$$

Substituting eqns. (A.2) and (A.12) in above,

$$\therefore \tan(\delta°) = \frac{2at}{2at^2}$$

$$\therefore \tan(\delta°) = \frac{1}{t}$$

$$\therefore t = \frac{1}{\tan(\delta°)} \tag{1.2}$$

Equating eqns. (1.1) & (1.2),

let, $\angle$ PFQ $= \theta°$

$$t = 2\cot(\varphi°) = \cot(\delta°) = \cot\left(\frac{\theta°}{2}\right) \tag{1.3}$$

Eqn. (1.3) is mathematical expression of the theorem.

THEOREM- 2:

Relation between $\boldsymbol{\theta}°$ & $\boldsymbol{t}$. In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$, then the t is called parameter of the parabola and its value is given with respect to $\boldsymbol{\theta}°$

$$\boldsymbol{t = \frac{1 + cos(\theta°)}{sin(\theta°)}}$$

Referring fig. 4,

$$\text{Eqn. (D.12)} \Rightarrow \tan(\theta°) = \frac{2t}{t^2 - 1} \tag{2.1}$$

Substituting eqns. (1.2) in above eqn.,

$$\therefore \tan(\theta°) = \frac{2\left(\frac{1}{\tan(\delta°)}\right)}{\left(\frac{1}{\tan(\delta°)}\right)^2 - 1}$$

$$\therefore \tan(\theta°) = \frac{\left(\frac{2}{\tan(\delta°)}\right)}{\left(\frac{1}{\tan^2(\delta°)}\right)^2 - 1}$$

$$\therefore \tan(\theta°) = \frac{\left(\frac{2}{\tan(\delta°)}\right)}{\left(\frac{1}{\tan^2(\delta°)} - 1\right)}$$

$$\therefore \tan(\theta°) = \frac{\left(\frac{2}{\tan(\delta°)}\right)}{\left(\frac{1 - \tan^2(\delta°)}{\tan^2(\delta°)}\right)}$$

$$\therefore \tan(\theta°) = \left(\frac{2}{\tan(\delta°)}\right) \times \left(\frac{\tan^2(\delta°)}{1 - \tan^2(\delta°)}\right)$$

$$\therefore \tan(\theta°) = \frac{2\tan(\delta°)}{1 - \tan^2(\delta°)}$$

$$\therefore \tan(\theta°) = \tan(2\delta°)$$

$$\therefore \theta° = 2\delta° \qquad (2.2)$$

$$\tan(\theta°) = \frac{PQ}{FQ}$$

Substituting eqn. (A.7) in above,

$$\therefore \tan(\theta°) = \frac{2at}{a(t^2 - 1)}$$

$$\therefore \tan(\theta°) = \frac{2t}{t^2 - 1}$$

$$\therefore (t^2 - 1)\tan(\theta°) = 2t$$

$$\therefore t^2\tan(\theta°) - \tan(\theta°) = 2t$$

$$\therefore t^2\tan(\theta°) - 2t - \tan(\theta°) = 0$$

Solving the above quadratic eqn.,

$$t = \frac{2 \pm \sqrt{4 + 4\tan^2(\theta°)}}{2\tan(\theta°)}$$

$$\therefore t = \frac{2 \pm 2\sqrt{1 + \tan^2(\theta°)}}{2\tan(\theta°)}$$

$$\therefore t = \frac{\left(1 \pm \sqrt{\frac{1}{\cos^2(\theta°)}}\right)}{\tan(\theta°)}$$

$$\therefore t = \frac{\left(1 \pm \frac{1}{\cos(\theta°)}\right)}{\frac{\sin(\theta°)}{\cos(\theta°)}}$$

$$\therefore t = \frac{\left(\frac{\cos(\theta°) \pm 1}{\cos(\theta°)}\right)}{\frac{\sin(\theta°)}{\cos(\theta°)}}$$

$$\therefore t = \left(\frac{\cos(\theta°) \pm 1}{\cos(\theta°)}\right) \times \left(\frac{\cos(\theta°)}{\sin(\theta°)}\right)$$

$$\therefore t = \frac{\cos(\theta°) \pm 1}{\sin(\theta°)}$$

$$\therefore t = \frac{\cos(\theta°) + 1}{\sin(\theta°)} \quad (or)\ t = \frac{\cos(\theta°) - 1}{\sin(\theta°)}$$

$$t = \frac{\cos(\theta°) - 1}{\sin(\theta°)} \text{ is invalid}$$

$$\therefore t = \frac{1 + \cos(\theta°)}{\sin(\theta°)} \quad (2.3)$$

Eqn. (2.2) & (2.3) is mathematical expression of the theorem.

THEOREM- 3:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{\frac{1}{FX^2} + \frac{1}{FY^2} = \frac{1}{2}\left(\frac{OB}{FC \times OF^2}\right)}$$

Derivations for proof of the theorem

$$\text{Eqn. (E. 1)} \Longrightarrow FX = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1} - 1}$$

$$\therefore FX^2 = \frac{4a^2(t^2+1)}{t^2 + 1 + 1 - 2\sqrt{t^2+1}}$$

$$= \frac{4a^2(t^2+1)}{t^2 + 2 - 2\sqrt{t^2+1}} \quad (3.1)$$

$$\text{Eqn. (E. 2)} \Longrightarrow \text{FY} = \frac{2\text{a}\sqrt{\text{t}^2+1}}{\sqrt{\text{t}^2+1}+1}$$

$$\therefore \text{FY}^2 = \frac{4\text{a}^2(\text{t}^2+1)}{\text{t}^2+1+1+2\sqrt{\text{t}^2+1}}$$
$$= \frac{4\text{a}^2(\text{t}^2+1)}{\text{t}^2+2+2\sqrt{\text{t}^2+1}} \quad (3.2)$$

Adding reciprocal of eqns. (3.1) & (3.2),

$$\frac{1}{\text{FX}^2}+\frac{1}{\text{FY}^2} = \left(\frac{\text{t}^2+2-2\sqrt{\text{t}^2+1}}{4\text{a}^2(\text{t}^2+1)}\right)+\left(\frac{\text{t}^2+2+2\sqrt{\text{t}^2+1}}{4\text{a}^2(\text{t}^2+1)}\right)$$

$$\therefore \frac{1}{\text{FX}^2}+\frac{1}{\text{FY}^2} = \frac{\text{t}^2+2-2\sqrt{\text{t}^2+1}+\text{t}^2+2+2\sqrt{\text{t}^2+1}}{4\text{a}^2(\text{t}^2+1)}$$

$$\therefore \frac{1}{\text{FX}^2}+\frac{1}{\text{FY}^2}$$
$$= \frac{2(\text{t}^2+2)}{4\text{a}^2(\text{t}^2+1)} \quad (3.3)$$

$$\therefore \frac{1}{\text{FX}^2}+\frac{1}{\text{FY}^2} = \frac{(\text{t}^2+2)}{2\text{a}^2(\text{t}^2+1)}$$

Referring fig. 3,

$$\text{Eqn. (C. 1)} \Longrightarrow \text{OB} = \frac{\text{at}\times(\text{t}^2+2)}{\sqrt{\text{t}^2+1}}$$

$$\text{Eqn. (C. 2)} \Longrightarrow \text{FC} = \text{at}\sqrt{\text{t}^2+1}$$

$$\frac{1}{2\times\text{OF}^2}\times\left(\frac{\text{OB}}{\text{FC}}\right) = \frac{1}{2\text{a}^2}\times\frac{\left(\frac{\text{at}\times(\text{t}^2+2)}{\sqrt{\text{t}^2+1}}\right)}{\text{at}\sqrt{\text{t}^2+1}}$$

$$\therefore \frac{1}{2\times\text{OF}^2}\times\frac{\text{OB}}{\text{FC}} = \frac{1}{2\text{a}^2}\times\frac{\text{at}\times(\text{t}^2+2)}{\sqrt{\text{t}^2+1}}\times\frac{1}{\text{at}\sqrt{\text{t}^2+1}}$$

$$\therefore \frac{1}{2}\times\left(\frac{\text{OB}}{\text{FC}\times\text{OF}^2}\right)$$
$$= \frac{(\text{t}^2+2)}{2\text{a}^2(\text{t}^2+1)} \quad (3.4)$$

Equating eqns. (3.3) & (3.4),

$$\frac{1}{FX^2} + \frac{1}{FY^2} = \frac{1}{2}\left(\frac{OB}{FC \times OF^2}\right) \quad (0.5)$$

Eqn. (3.5) is mathematical expression of the theorem.

THEOREM- 4:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FM = SC}$

Derivations for proof of the theorem

$$\text{Eqn.}(A.18) \Longrightarrow FM = a\sqrt{t^2 + 1} \quad (4.1)$$

$$\text{Eqn.}(C.9) \Longrightarrow SC = a\sqrt{t^2 + 1} \quad (4.2)$$

Equating the eqns. (4.1) & (4.2),

$$FM = SC \quad (4.3)$$

Eqn. (4.3) is mathematical expression of the theorem.

THEOREM- 5:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FQ = TA}$

Derivations for proof of the theorem

$$\text{Eqn.}(A.7) \Longrightarrow FQ = a(t^2 - 1) \quad (5.1)$$

$$\text{Eqn.}(A.21) \Longrightarrow TA = a(t^2 - 1) \quad (5.2)$$

Equating the eqns. (5.1) & (5.2),

$$FQ = TA \quad (5.3)$$

Eqn. (5.3) is mathematical expression of the theorem.

THEOREM- 6:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FN = FZ}$

Derivations for proof of the theorem

$$\text{Eqn. (B.2)} \Longrightarrow \text{FN} = \frac{a(t^2+1)}{t} \tag{6.1}$$

$$\text{Eqn. (B.10)} \Longrightarrow \text{FZ} = \frac{a(t^2+1)}{t} \tag{6.2}$$

Equating the eqns. (6.1) & (6.2),

$$\text{FN} = \text{FZ} \tag{6.3}$$

Eqn. (6.3) is mathematical expression of the theorem.

THEOREM- 7:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{J\mu = \mu\lambda}$

Derivations for proof of the theorem

$$\text{Eqn. (E.8)} \Longrightarrow \text{J}\mu = \frac{a\sqrt{t^2+1}}{t^2} \tag{7.1}$$

$$\text{Eqn. (E.20)} \Longrightarrow \mu\lambda = \frac{a\sqrt{t^2+1}}{t^2} \tag{7.2}$$

Equating eqns. (7.1) & (7.2),

$$\text{J}\mu = \mu\lambda \tag{7.3}$$

Eqn. (7.3) is mathematical expression of the theorem.

THEOREM- 8:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{RK = GH}$

Derivations for proof of the theorem

$$\text{Eqn. (C. 5)} \Longrightarrow \text{RK} = 4\text{a} \tag{8.1}$$

$\text{Eqn. (A. 5)} \Longrightarrow \text{FG} = \text{FH} = 2\text{a}$

$$\therefore \text{GH} = 2 \times \text{FG} = 4\text{a} \tag{8.2}$$

Equating eqns. (8.1), (8.2)

$$\text{RK} = \text{GH} \tag{8.3}$$

Eqn. (8.3) is mathematical expression of the theorem.

THEOREM- 9:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{F}\lambda = \boldsymbol{FJ}$

Derivations for proof of the theorem

$$\text{Eqn. (E. 16)} \Longrightarrow \text{F}\lambda = \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2} \tag{9.1}$$

$$\text{Eqn. (D. 5)} \Longrightarrow \text{FJ} = \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2} \tag{9.2}$$

Equating the eqns. (9.1) & (9.2),

$$\text{F}\lambda = \text{FJ} \tag{9.3}$$

Eqn. (9.3) is mathematical expression of the theorem.

THEOREM- 10:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TL = TN}$

Derivations for proof of the theorem

$$\text{Eqn. (E. 16)} \Longrightarrow \text{TL} = \frac{\text{a}(\text{t}^2 - 1)\sqrt{\text{t}^2 + 1}}{\text{t}} \tag{10.1}$$

$$\text{Eqn.}\,(\text{B}.3) \Longrightarrow \text{TN} = \frac{\text{a}(\text{t}^2-1)\sqrt{\text{t}^2+1}}{\text{t}} \qquad (10.2)$$

Equating the eqns. (10.1) & (10.2),

$$\text{TL} = \text{TN} \qquad (10.3)$$

Eqn. (10.3) is mathematical expression of the theorem.

THEOREM- 11:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{Q}\lambda = \boldsymbol{TW}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{E}.17) \Longrightarrow \text{Q}\lambda = \frac{\text{a}(\text{t}^4+1)}{\text{t}^2} \qquad (11.1)$$

$$\text{Eqn.}\,(\text{D}.13) \Longrightarrow \text{TW} = \frac{\text{a}(\text{t}^4+1)}{\text{t}^2} \qquad (11.2)$$

Equating the eqns. (11.1) & (11.2),

$$\text{Q}\lambda = \text{TW} \qquad (11.3)$$

Eqn. (11.3) is mathematical expression of the theorem.

THEOREM- 12:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{S}\lambda = \boldsymbol{PJ}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{E}.18) \Longrightarrow \text{S}\lambda = \frac{\text{a}(\text{t}^2+1)^2}{\text{t}^2} \qquad (12.1)$$

$$\text{Eqn.}\,(\text{D}.6) \Longrightarrow \text{PJ} = \frac{\text{a}(\text{t}^2+1)^2}{\text{t}^2} \qquad (12.2)$$

Equating the eqns. (12.1) & (12.2),

$$S\lambda = \mathrm{PJ} \tag{12.3}$$

Eqn. (12.3) is mathematical expression of the theorem.

THEOREM- 13:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{FN = FZ}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.2) \Longrightarrow \mathrm{FN} = \frac{\mathrm{a}(\mathrm{t}^2+1)}{\mathrm{t}} \tag{13.1}$$

$$\text{Eqn.}\,(\text{B}.10) \Longrightarrow \mathrm{FZ} = \frac{\mathrm{a}(\mathrm{t}^2+1)}{\mathrm{t}} \tag{13.2}$$

Equating eqns. (13.1), (13.2)

$$\mathrm{FN} = \mathrm{FZ} \tag{13.3}$$

Eqn. (13.3) is mathematical expression of the theorem.

THEOREM- 14:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{OV = PB}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.7) \Longrightarrow \mathrm{OV} = \frac{\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}} \tag{14.1}$$

$$\text{Eqn.}\,(\text{C}.13) \Longrightarrow \mathrm{PB} = \frac{\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}} \tag{14.2}$$

Equating eqns. (14.1), (14.2)

$$\mathrm{OV} = \mathrm{PB} \tag{14.3}$$

Eqn. (14.3) is mathematical expression of the theorem.

THEOREM- 15:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{AQ = FS}$

Derivations for proof of the theorem

$$\text{Eqn. (A.29)} \Longrightarrow \text{AQ} = \text{a}(\text{t}^2 + 1) \tag{15.1}$$

$$\text{Eqn. (A.23)} \Longrightarrow \text{FS} = \text{a}(\text{t}^2 + 1) \tag{15.2}$$

Equating eqns. (15.1), (15.2)

$$\text{AQ} = \text{FS} \tag{15.3}$$

Eqn. (15.3) is mathematical expression of the theorem.

THEOREM- 16:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{JN = P\Omega}$

Derivations for proof of the theorem

$$\text{Eqn. (B.8)} \Longrightarrow \text{JN} = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}^2} \tag{16.1}$$

$$\text{Eqn. (A.23)} \Longrightarrow \text{P}\Omega = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}^2} \tag{16.2}$$

Equating eqns. (16.1), (16.2)

$$\text{JN} = \text{P}\Omega \tag{16.3}$$

Eqn. (16.3) is mathematical expression of the theorem.

THEOREM- 17:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TN = TL}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.3) \Longrightarrow \text{TN} = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \tag{17.1}$$

$$\text{Eqn.}\,(\text{E}.26) \Longrightarrow \text{TL} = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \tag{17.2}$$

Equating eqns. (17.1), (17.2)

$$\text{TN} = \text{TL} \tag{17.3}$$

Eqn. (17.3) is mathematical expression of the theorem.

THEOREM- 18:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FZ = M\mu}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.10) \Longrightarrow \text{FZ} = \frac{a(t^2+1)}{t} \tag{18.1}$$

$$\text{Eqn.}\,(\text{E}.5) \Longrightarrow \text{M}\mu = \frac{a(t^2+1)}{t} \tag{18.2}$$

Equating eqns. (18.1), (18.2)

$$\text{FZ} = \text{M}\mu \tag{18.3}$$

Eqn. (18.3) is mathematical expression of the theorem.

THEOREM- 19:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{SC = FM = N\mu}$

Derivations for proof of the theorem

$$\text{Eqn. (C.9)} \Rightarrow \text{SC} = a\sqrt{t^2+1} \quad (19.1)$$

$$\text{Eqn. (A.18)} \Rightarrow \text{FM} = a\sqrt{t^2+1} \quad (19.2)$$

$$\text{Eqn. (E.7)} \Rightarrow \text{N}\mu = a\sqrt{t^2+1} \quad (19.3)$$

Equating eqns. (19.1), (19.2) & (19.3)

$$\text{SC} = \text{N}\mu \quad (19.4)$$

Eqn. (19.4) is mathematical expression of the theorem.

THEOREM- 20:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{SZ = PN = PL = J\Omega}$

Derivations for proof of the theorem

$$\text{Eqn. (D.22)} \Rightarrow \text{SZ} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (20.1)$$

$$\text{Eqn. (B.4)} \Rightarrow \text{PN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (20.2)$$

$$\text{Eqn. (E.27)} \Rightarrow \text{PL} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (20.3)$$

$$\text{Eqn. (E.9)} \Rightarrow \text{J}\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (20.4)$$

Equating the eqns. (20.1), (20.2), (20.3) & (20.4),

$$\text{SZ} = \text{PN} = \text{PL} = \text{J}\Omega \quad (20.5)$$

Eqn. (20.5) is mathematical expression of the theorem.

THEOREM- 21:

In a parabola $\boldsymbol{y^2} = \boldsymbol{4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{W}\lambda = \boldsymbol{2} \times \boldsymbol{O}\lambda$

Derivations for proof of the theorem

$$\text{Eqn. (E. 15)} \Longrightarrow \text{W}\lambda = \frac{2a}{t^2} \tag{21.1}$$

$\text{Eqn. (E. 14)} \Longrightarrow \text{O}\lambda = \frac{a}{t^2}$

Multiplying above eqn. by 2

$$\therefore 2 \times \text{O}\lambda = \frac{2a}{t^2} \tag{21.2}$$

Equating the eqns. (21.1) & (21.2),

$$\text{W}\lambda = 2 \times \text{O}\lambda \tag{21.3}$$

Eqn. (21.3) is mathematical expression of the theorem.

THEOREM- 22:

In a parabola $\boldsymbol{y^2} = \boldsymbol{4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{M\sigma} = \frac{\boldsymbol{PQ}}{\boldsymbol{2}}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn. (E. 21)} \Longrightarrow \text{M}\sigma = \text{at} \tag{22.1}$$

$\text{Eqn. (D. 13)} \Longrightarrow \text{PQ} = 2\text{at}$

Dividing above eqn. by 2

$$\therefore \frac{\text{PQ}}{2} = \text{at} \tag{22.2}$$

Equating the eqns. (22.1) & (22.2),

$$M\sigma = \frac{PQ}{2} \tag{22.3}$$

Eqn. (22.3) is mathematical expression of the theorem.

THEOREM- 23:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TQ = 2 \times OQ}$

Derivations for proof of the theorem

$$\text{Eqn. (A. 12)} \Longrightarrow TQ = 2at^2 \tag{23.1}$$

$\text{Eqn. (A. 1)} \Longrightarrow OQ = at^2$

Multiplying above eqn. by 2,

$$2 \times OQ = 2at^2 \tag{23.2}$$

Equating the above eqns. (23.1) & (23.2),

$$TQ = 2 \times OQ \tag{23.3}$$

Eqn. (23.3) is mathematical expression of the theorem.

THEOREM- 24:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PS = 2 \times SC}$

Derivations for proof of the theorem

$$\text{Eqn. (A. 14)} \Longrightarrow PS = 2a\sqrt{t^2 + 1} \tag{24.1}$$

$\text{Eqn. (C. 9)} \Longrightarrow SC = a\sqrt{t^2 + 1}$

Multiplying above eqn. by 2,

$$2 \times SC = 2a\sqrt{t^2 + 1} \tag{24.2}$$

Equating the above eqns. (24.1) & (24.2),

$$PS = 2 \times SC \tag{24.3}$$

Eqn. (24.3) is mathematical expression of the theorem.

THEOREM- 25:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{PQ = 2 \times OM}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(A.2) \Longrightarrow PQ = 2at \tag{25.1}$$

$$\text{Eqn.}\,(A.17) \Longrightarrow OM = at \tag{25.2}$$

Multiplying eqns. (25.2) by 2,

$2 \times OM = 2 \times at$

Equating the above eqn. & eqn. (25.1)

$$PQ = 2 \times OM \tag{25.3}$$

Eqn. (25.3) is mathematical expression of the theorem.

THEOREM- 26:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{TS = 2 \times FP}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(A.15) \Longrightarrow TS = 2a(t^2 + 1) \tag{26.1}$$

$\text{Eqn.}\,(A.9) \Longrightarrow FP = a(t^2 + 1)$

$$\therefore 2 \times FP = 2a(t^2 + 1) \tag{26.2}$$

Equating eqns. (26.1) & (26.2),

$$\mathrm{TS} = 2 \times \mathrm{FP} \quad (26.3)$$

Eqn. (26.3) is mathematical expression of the theorem.

THEOREM- 27:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{UQ = 2 \times OV}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.5) \Longrightarrow \mathrm{UQ} = \frac{2\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}} \quad (27.1)$$

$$\text{Eqn.}\,(\text{A}.7) \Longrightarrow \mathrm{OV} = \frac{\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}}$$

$$\therefore 2 \times \mathrm{OV} = \frac{\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}} \quad (27.2)$$

Equating eqns. (27.1), (27.2)

$$\mathrm{UQ} = 2 \times \mathrm{OV} \quad (27.3)$$

Eqn. (27.3) is mathematical expression of the theorem.

THEOREM- 28:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{UQ = 2 \times OV}$

Derivations for proof of the theorem

$$\text{Eqn.}\,(\text{B}.5) \Longrightarrow \mathrm{PQ} = 2\mathrm{at} \quad (28.1)$$

$$\text{Eqn.}\,(\text{E}.21) \Longrightarrow \mathrm{M\sigma} = \mathrm{at}$$

$$\therefore 2 \times M\sigma = 2at \tag{28.2}$$

Equating eqns. (28.1), (28.2)

$$PQ = 2 \times M\sigma \tag{28.3}$$

Eqn. (28.3) is mathematical expression of the theorem.

THEOREM- 29:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $\boldsymbol{(x, y) = (at^2, 2at)}$,

$\boldsymbol{PT = 2 \times QM = 2 \times PM}$

Derivations for proof of the theorem

$$\text{Eqn. (A.13)} \Longrightarrow PT = 2at\sqrt{t^2 + 1} \tag{29.1}$$

$$\text{Eqn. (A.22)} \Longrightarrow QM = at\sqrt{t^2 + 1}$$

$$\therefore 2 \times QM = at\sqrt{t^2 + 1} \tag{29.2}$$

$$\text{Eqn. (A.20)} \Longrightarrow PM = at\sqrt{t^2 + 1}$$

$$\therefore 2 \times QM = at\sqrt{t^2 + 1} \tag{29.3}$$

Equating eqns. (29.1), (29.2) & (29.3),

$$PT = 2 \times QM = 2 \times PM \tag{29.4}$$

Eqn. (29.4) is mathematical expression of the theorem.

THEOREM- 30:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $\boldsymbol{(x, y) = (at^2, 2at)}$,

$\boldsymbol{SQ = 2(OS - FP) = 2a}$

Derivations for proof of the theorem

$$\text{Eqn. (A.6)} \Longrightarrow SQ = 2a \tag{30.1}$$

$$\text{Eqn. (A. 10)} \Longrightarrow \text{OS} = \text{a}(\text{t}^2 + 2) \quad (30.2)$$

$$\text{Eqn. (A. 5)} \Longrightarrow \text{FP} = \text{a}(\text{t}^2 + 1) \quad (30.3)$$

Subtracting eqns. (30.3) from (30.2),

$\text{OS} - \text{FP} = \text{a}(\text{t}^2 + 2) - \text{a}(\text{t}^2 + 1)$

$\therefore \text{OS} - \text{FP} = (\text{at}^2 + 2\text{a}) - (\text{at}^2 + \text{a})$

$\therefore \text{OS} - \text{FP} = \text{at}^2 + 2\text{a} - \text{at}^2 - \text{a}$

$\therefore \text{OS} - \text{FP} = \text{a}$

$$\therefore 2(\text{OS} - \text{FP}) = 2\text{a} \quad (30.4)$$

Equating eqns. (30.1) & (30.4),

$$\text{SQ} = 2(\text{OS} - \text{FP}) = 2\text{a} \quad (30.5)$$

Eqn. (30.5) is mathematical expression of the theorem.

THEOREM- 31:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{XY = 4 \times FJ}$

Derivations for proof of the theorem

$$\text{Eqn. (E. 4)} \Longrightarrow \text{XY} = \frac{4\text{a}(\text{t}^2 + 1)}{\text{t}^2} \quad (31.1)$$

$$\text{Eqn. (D. 5)} \Longrightarrow \text{FJ} = \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2}$$

Multiplying above eqn. by 4,

$$4 \times \text{FJ} = 4 \times \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2} \quad (31.2)$$

Equating the above eqns. (31.1) & (31.2),

$$XY = 4 \times FJ \tag{31.3}$$

Eqn. (31.3) is mathematical expression of the theorem.

THEOREM- 32:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{OT \times OW = OF}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn.}\,(A.11) \Longrightarrow OT = at^2 \tag{32.1}$$

$$\text{Eqn.}\,(D.11) \Longrightarrow OW = \frac{1}{t^2} \tag{32.2}$$

$$\text{Eqn.}\,(A.4) \Longrightarrow OF = a \tag{32.3}$$

Multiplying eqns. (32.1) & (32.2)

$$OT \times OW = at^2 \times \frac{1}{t^2}$$

$$\therefore OT \times OW = a \tag{32.4}$$

Equating the eqns. (32.4) & (32.3),

$$OT \times OW = OF \tag{32.5}$$

Eqn. (32.5) is mathematical expression of the theorem.

THEOREM- 33:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PN \times NM = FZ}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn. (B.4)} \Longrightarrow \text{PN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (33.1)$$

$$\text{Eqn. (D.6)} \Longrightarrow \text{NM} = \frac{a\sqrt{t^2+1}}{t} \quad (33.2)$$

$$\text{Eqn. (B.10)} \Longrightarrow \text{FZ} = \frac{a(t^2+1)}{t}$$

$$\text{FZ}^2 = \left(\frac{a(t^2+1)}{t}\right)^2 \quad (33.3)$$

Multiplying eqns. (33.1) & (33.2)

$$\text{PN} \times \text{NM} = \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t}\right) \times \left(\frac{a\sqrt{t^2+1}}{t}\right)$$

$$\therefore \text{PN} \times \text{NM} = \frac{a^2(t^2+1)^2}{t^2}$$

$$\therefore \text{PN} \times \text{NM} = \left(\frac{a(t^2+1)}{t}\right)^2 \quad (33.4)$$

Equating the eqns. (33.4) & (33.3),

$$\text{PN} \times \text{NM} = \text{FZ}^2 \quad (33.5)$$

Eqn. (33.5) is mathematical expression of the theorem.

THEOREM- 34:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PQ \times OW = JW}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (A.2)} \Longrightarrow \text{PQ} = 2\text{at} \quad (34.1)$$

$$\text{Eqn. (D. 11)} \Longrightarrow \text{OW} = \frac{1}{t^2} \quad (34.2)$$

$$\text{Eqn. (D. 8)} \Longrightarrow \text{JW} = \frac{2a}{t} \quad (34.3)$$

Multiplying eqns. (34.1) & (34.2)

$$\text{PQ} \times \text{OW} = 2at \times \left(\frac{1}{t^2}\right)$$

$$\therefore \text{PQ} \times \text{OW} = \frac{2a}{t} \quad (34.4)$$

Equating the eqns. (34.4) & (34.3),

$$\text{PQ} \times \text{OW} = \text{JW} \quad (34.5)$$

Eqn. (34.5) is mathematical expression of the theorem.

THEOREM- 35:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{M\sigma \times \partial\sigma = OQ^2}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn. (E. 17)} \Longrightarrow \text{M}\sigma = \text{at} \quad (35.1)$$

$$\text{Eqn. (E. 22)} \Longrightarrow \partial\sigma = \text{at}^3 \quad (35.2)$$

$$\text{Eqn. (A. 1)} \Longrightarrow \text{OQ} = \text{at}^2$$

$$〖\therefore \text{OQ}〗 \text{^}2 = \text{a^}2 \text{ t^}4 \quad (35.3)$$

Multiplying eqns. (35.1) & (35.2)

$$\text{M}\sigma \times \partial\sigma = \text{at} \times \text{at}^3$$

$$\therefore \text{M}\sigma \times \partial\sigma = \text{a}^2\text{t}^4 \quad (35.4)$$

Equating the eqns. (35.4) & (35.3),

$$M\sigma \times \partial\sigma = OQ^2 \quad (35.5)$$

Eqn. (35.5) is mathematical expression of the theorem.

THEOREM- 36:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PS \times PD = PT \times QD = PQ^2}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn.}\,(A.14) \Rightarrow PS = 2a\sqrt{t^2+1} \quad (36.1)$$

$$\text{Eqn.}\,(D.20) \Rightarrow PD = \frac{2at^2}{\sqrt{t^2+1}} \quad (36.2)$$

$$\text{Eqn.}\,(A.13) \Rightarrow PT = 2at\sqrt{t^2+1} \quad (36.3)$$

$$\text{Eqn.}\,(C.3) \Rightarrow QD = \frac{2at}{\sqrt{t^2+1}} \quad (36.4)$$

$$\text{Eqn.}\,(A.1) \Rightarrow PQ = 2at$$

$$\therefore PQ^2 = 4a^2t^2 \quad (36.5)$$

Multiplying eqns. (36.1) & (36.2)

$$PS \times PD = 2a\sqrt{t^2+1} \times \frac{2at^2}{\sqrt{t^2+1}}$$

$$\therefore PS \times PD = 4a^2t^2 \quad (36.6)$$

Multiplying eqns. (36.3) & (36.4)

$$PT \times QD = 2at\sqrt{t^2+1} \times \frac{2at}{\sqrt{t^2+1}}$$

$$\therefore PT \times QD = 4a^2t^2 \quad (36.7)$$

Equating the eqns. (36.5), (36.6) & (36.7),

$$PS \times PD = PT \times QD = PQ^2 \quad (36.8)$$

Eqn. (36.8) is mathematical expression of the theorem.

THEOREM- 37:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{QM} \times \boldsymbol{SZ} = \boldsymbol{J}\Omega \times \boldsymbol{SZ} = \boldsymbol{PQ^2}$

Derivations for proof of the theorem

Referring fig. 8,

$$\text{Eqn. (A.22)} \Longrightarrow QM = at\sqrt{t^2+1} \quad (37.1)$$

$$\text{Eqn. (A.19)} \Longrightarrow TM = at\sqrt{t^2+1} \quad (37.2)$$

$$\text{Eqn. (A.20)} \Longrightarrow PM = at\sqrt{t^2+1} \quad (37.3)$$

$$\text{Eqn. (D.17)} \Longrightarrow SZ = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (37.4)$$

$$\text{Eqn. (E.9)} \Longrightarrow J\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (37.5)$$

$$\text{Eqn. (A.9)} \Longrightarrow FP = a(t^2+1)$$

$$\therefore FP^2 = a^2(t^2+1)^2 \quad (37.6)$$

$$\text{Eqn. (A.16)} \Longrightarrow FT = a(t^2+1)$$

$$\therefore FT^2 = a^2(t^2+1)^2 \quad (37.7)$$

$\text{Eqn.}(A.23) \Longrightarrow FS = a(t^2 + 1)$

$$\therefore FS^2 = a^2(t^2 + 1)^2 \quad (37.8)$$

Multiplying eqns. (37.1) & (37.2)

$$QM \times SZ = at\sqrt{t^2 + 1} \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore QM \times SZ = a^2(t^2 + 1)^2 \quad (37.9)$$

Multiplying eqns. (37.3) & (37.4)

$$QM \times J\Omega = at\sqrt{t^2 + 1} \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore QM \times J\Omega = a^2(t^2 + 1)^2 \quad (37.10)$$

Multiplying eqns. (37.3) & (37.4)

$$TM \times SZ = at\sqrt{t^2 + 1} \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore TM \times SZ = a^2(t^2 + 1)^2 \quad (37.11)$$

Multiplying eqns. (37.3) & (37.4)

$$TM \times J\Omega = at\sqrt{t^2 + 1} \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore TM \times J\Omega = a^2(t^2 + 1)^2 \quad (37.12)$$

Multiplying eqns. (37.3) & (37.4)

$$PM \times SZ = at\sqrt{t^2 + 1} \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore PM \times SZ = a^2(t^2 + 1)^2 \quad (37.13)$$

Multiplying eqns. (37.3) & (37.4)

$$PM \times J\Omega = at\sqrt{t^2+1} \times \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t}\right)$$

$$\therefore PM \times J\Omega = a^2(t^2+1)^2 \quad (37.14)$$

Equating eqns. (37.9), (37.10) & (37.11), (37.12), (37.13), (37.14), (37.6), (37.7) & (37.8),

$$QM \times SZ = QM \times J\Omega = TM \times SZ = TM \times J\Omega = PM \times SZ = PM \times J\Omega = FP^2 = FT^2 = FS^2 \quad (37.15)$$

Eqn. (37.15) is mathematical expression of the theorem.

THEOREM- 38:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{O}\sigma \times \partial\sigma = \boldsymbol{OT} \times \boldsymbol{TQ} = \boldsymbol{2} \times (\boldsymbol{OQ})^{\boldsymbol{2}}$$

Derivations for proof of the theorem

Referring fig. 7,

$$\text{Eqn.}\,(A.2) \Rightarrow PQ = 2at \quad (38.1)$$

$$\text{Eqn.}\,(E.22) \Rightarrow \partial\sigma = at^3 \quad (38.2)$$

$$\text{Eqn.}\,(A.11) \Rightarrow OT = at^2 \quad (38.3)$$

$$\text{Eqn.}\,(A.12) \Rightarrow TQ = 2at^2 \quad (38.4)$$

$\text{Eqn.}\,(A.1) \Rightarrow OQ = at^2$

$$\therefore 2 \times (OQ)^2 = 2a^2t^4 \quad (38.5)$$

Multiplying eqns. (38.1) & (38.2)

$PQ \times \partial\sigma = 2at \times at^3$

Referring fig. 7, $O\sigma = PQ$

$$\therefore O\sigma \times \partial\sigma = 2a^2t^4 \quad (38.6)$$

Multiplying eqns. (38.3) & (38.4)

$\mathrm{OT} \times \mathrm{TQ} = \mathrm{at}^2 \times 2\mathrm{at}^2$

$$\therefore \mathrm{OT} \times \mathrm{TQ} = 2\mathrm{a}^2\mathrm{t}^4 \qquad (38.7)$$

Equating the eqns. (38.6), (38.7) & (38.5),

$$\mathrm{O}\sigma \times \partial\sigma = \mathrm{OT} \times \mathrm{TQ} = 2 \times (\mathrm{OQ})^2 \qquad (38.8)$$

Eqn. (38.8) is mathematical expression of the theorem.

THEOREM- 39:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PQ \times JW = TQ \times W\lambda}$

Derivations for proof of the theorem

Referring fig. 8,

$$\mathrm{Eqn.}\,(\mathrm{A}.2) \Longrightarrow \mathrm{PQ} = 2\mathrm{at} \qquad (39.1)$$

$$\mathrm{Eqn.}\,(\mathrm{D}.8) \Longrightarrow \mathrm{JW} = \frac{2\mathrm{a}}{\mathrm{t}} \qquad (39.2)$$

$$\mathrm{Eqn.}\,(\mathrm{A}.12) \Longrightarrow \mathrm{TQ} = 2\mathrm{at}^2 \qquad (39.3)$$

$$\mathrm{Eqn.}\,(\mathrm{E}.15) \Longrightarrow \mathrm{W}\lambda = \frac{2\mathrm{a}}{\mathrm{t}^2} \qquad (39.4)$$

Multiplying eqns. (39.1) & (39.2)

$$\mathrm{PQ} \times \mathrm{JW} = 2\mathrm{at} \times \frac{2\mathrm{a}}{\mathrm{t}}$$

$$\therefore \mathrm{PQ} \times \mathrm{JW} = 4\mathrm{a}^2 \qquad (39.5)$$

Multiplying eqns. (39.3) & (39.4)

$$\mathrm{TQ} \times \mathrm{W}\lambda = 2\mathrm{at}^2 \times \frac{2\mathrm{a}}{\mathrm{t}^2}$$

$$\therefore TQ \times W\lambda = 4a^2 \quad (39.6)$$

Equating the eqns. (39.5) & (39.6),

$$PQ \times JW = TQ \times W\lambda \quad (39.7)$$

Eqn. (39.7) is mathematical expression of the theorem.

THEOREM- 40:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{OQ} \times \boldsymbol{PQ} = \boldsymbol{PB} \times \boldsymbol{PT}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (A.1)} \Rightarrow OQ = at^2 \quad (40.1)$$

$$\text{Eqn. (A.2)} \Rightarrow PQ = 2at \quad (40.2)$$

$$\text{Eqn. (C.13)} \Rightarrow PB = \frac{at^2}{\sqrt{t^2+1}} \quad (40.3)$$

$$\text{Eqn. (A.13)} \Rightarrow PT = 2at\sqrt{t^2+1} \quad (40.4)$$

Multiplying eqns. (40.1) & (40.2)

$$OQ \times PQ = at^2 \times 2at$$

$$\therefore OQ \times PQ = 2a^2t^3 \quad (40.5)$$

Multiplying eqns. (40.3) & (40.4)

$$PB \times PT = \frac{at^2}{\sqrt{t^2+1}} \times 2at\sqrt{t^2+1}$$

$$\therefore TQ \times W\lambda = 2a^2t^3 \quad (40.6)$$

Equating the eqns. (40.5) & (40.6),

$$OQ \times PQ = PB \times PT \quad (40.7)$$

Eqn. (40.7) is mathematical expression of the theorem.

THEOREM- 41:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{PR \times O\mu = PQ \times JW}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. }(A.25) \Longrightarrow PR = 4at \quad (41.1)$$

$$\text{Eqn. }(E.6) \Longrightarrow O\mu = \frac{a}{t} \quad (41.2)$$

$$\text{Eqn. }(A.2) \Longrightarrow PQ = 2at \quad (41.3)$$

$$\text{Eqn. }(D.8) \Longrightarrow JW = \frac{2a}{t} \quad (41.4)$$

$$\text{Eqn. }(B.5) \Longrightarrow UQ = \frac{2at^2}{\sqrt{t^2+1}} \quad (41.5)$$

$$\text{Eqn. }(E.19) \Longrightarrow J\lambda = \frac{2a\sqrt{t^2+1}}{t^2} \quad (41.6)$$

Multiplying eqns. (41.1) & (41.2)

$$PR \times O\mu = 4at \times \frac{a}{t}$$

$$\therefore PR \times O\mu = 4a^2 \quad (41.7)$$

Multiplying eqns. (41.3) & (41.4)

$$PQ \times JW = 2at \times \frac{2a}{t}$$

$$\therefore PQ \times JW = 4a^2 \tag{41.8}$$

Multiplying eqns. (41.5) & (41.6)

$$UQ \times J\lambda = \frac{2at^2}{\sqrt{t^2+1}} \times \frac{2a\sqrt{t^2+1}}{t^2}$$

$$\therefore UQ \times J\lambda = 4a^2 \tag{41.9}$$

Equating the eqns. (41.7), (41.8) & (41.9),

$$PR \times O\mu = PQ \times JW = UQ \times J\lambda \tag{41.10}$$

Eqn. (41.10) is mathematical expression of the theorem.

THEOREM- 42:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{\tau\omega \times F\omega = FQ \times TQ}$

Derivations for proof of the theorem

Referring fig. 7,

$$\text{Eqn. (E. 23)} \Longrightarrow \tau\omega = at(t^2 - 1) \tag{42.1}$$

$$\text{Eqn. (E. 24)} \Longrightarrow F\omega = 2at \tag{42.2}$$

$$\text{Eqn. (A. 7)} \Longrightarrow FQ = a(t^2 - 1) \tag{42.3}$$

$$\text{Eqn. (A. 12)} \Longrightarrow TQ = 2at^2 \tag{42.4}$$

Multiplying eqns. (42.1) & (42.2)

$$\tau\omega \times F\omega = at(t^2 - 1) \times 2at$$

$$\therefore \tau\omega \times F\omega = 2a^2t^2(t^2 - 1) \tag{42.5}$$

Multiplying eqns. (42.3) & (42.4)

$$FQ \times TQ = a(t^2 - 1) \times 2at^2$$

$$\therefore FQ \times TQ = 2a^2t^2(t^2 - 1) \qquad (42.6)$$

Equating the eqns. (42.5) & (42.6),

$$\tau\omega \times F\omega = FQ \times TQ \qquad (42.7)$$

Eqn. (42.7) is mathematical expression of the theorem.

THEOREM- 43:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$$\boldsymbol{TQ \times SQ = UQ \times PS = PT \times PU = PQ^2}$$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (A. 2)} \Longrightarrow PQ = 2at$$

$$\therefore PQ^2 = 4a^2t^2 \qquad (43.1)$$

$$\text{Eqn. (A. 12)} \Longrightarrow TQ = 2at^2 \qquad (43.2)$$

$$\text{Eqn. (A. 6)} \Longrightarrow SQ = 2a \qquad (43.3)$$

$$\text{Eqn. (B. 5)} \Longrightarrow UQ = \frac{2at^2}{\sqrt{t^2 + 1}} \qquad (43.4)$$

$$\text{Eqn. (A. 14)} \Longrightarrow PS = 2a\sqrt{t^2 + 1} \qquad (43.5)$$

$$\text{Eqn. (A. 13)} \Longrightarrow PT = 2at\sqrt{t^2 + 1} \qquad (43.6)$$

$$\text{Eqn. (B. 6)} \Longrightarrow PU = \frac{2at}{\sqrt{t^2 + 1}} \qquad (43.7)$$

Multiplying eqns. (0.2) & (0.3)

$$\mathrm{TQ} \times \mathrm{SQ} = 2\mathrm{at}^2 \times 2\mathrm{a}$$

$$\therefore \mathrm{TQ} \times \mathrm{SQ} = 4\mathrm{a}^2\mathrm{t}^2 \qquad (43.8)$$

Multiplying eqns. (0.4) & (0.5),

$$\mathrm{UQ} \times \mathrm{PS} = \frac{2\mathrm{at}^2}{\sqrt{\mathrm{t}^2+1}} \times 2\mathrm{a}\sqrt{\mathrm{t}^2+1}$$

$$\therefore \mathrm{UQ} \times \mathrm{PS} = 4\mathrm{a}^2\mathrm{t}^2 \qquad (43.9)$$

Multiplying eqns. (0.6) & (0.7),

$$\mathrm{PT} \times \mathrm{PU} = 2\mathrm{at}\sqrt{\mathrm{t}^2+1} \times \frac{2\mathrm{at}}{\sqrt{\mathrm{t}^2+1}}$$

$$\therefore \mathrm{PT} \times \mathrm{PU} = 4\mathrm{a}^2\mathrm{t}^2$$

Equating eqns. (0.1), (0.6) & (0.7),

$$\mathrm{TQ} \times \mathrm{SQ} = \mathrm{UQ} \times \mathrm{PS} = \mathrm{PT} \times \mathrm{PU} = \mathrm{PQ}^2 \qquad (43.10)$$

Eqn. (43.10) is mathematical expression of the theorem.

THEOREM- 44:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FP} \times \boldsymbol{FJ} = \mathbf{AQ} \times \boldsymbol{F\lambda} = \boldsymbol{FN^2}$

Derivations for proof of the theorem

Referring fig. 6,

$$\mathrm{Eqn.\,(A.9)} \Longrightarrow \mathrm{FP} = \mathrm{a}(\mathrm{t}^2+1) \qquad (44.1)$$

$$\mathrm{Eqn.\,(D.5)} \Longrightarrow \mathrm{FJ} = \frac{\mathrm{a}(\mathrm{t}^2+1)}{\mathrm{t}^2} \qquad (44.2)$$

$$\mathrm{Eqn.\,(A.29)} \Longrightarrow \mathrm{AQ} = \mathrm{a}(\mathrm{t}^2+1) \qquad (44.3)$$

$$\text{Eqn. (E.16)} \Longrightarrow \mathrm{F}\lambda = \frac{a(t^2+1)}{t^2} \qquad (44.4)$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{FN} = \frac{a(t^2+1)}{t}$$

$$\therefore \mathrm{FN}^2 = \frac{a^2(t^2+1)^2}{t^2} \qquad (44.5)$$

Multiplying eqns. (0.1) & (0.2)

$$\mathrm{FP} \times \mathrm{FJ} = a(t^2+1) \times \frac{a(t^2+1)}{t^2}$$

$$\therefore \mathrm{FP} \times \mathrm{FJ} = \frac{a^2(t^2+1)^2}{t^2} \qquad (44.6)$$

Multiplying eqns. (0.3) & (0.4)

$$\mathrm{AQ} \times \mathrm{F}\lambda = a(t^2+1) \times \frac{a(t^2+1)}{t^2}$$

$$\therefore \mathrm{AQ} \times \mathrm{F}\lambda = \frac{a^2(t^2+1)^2}{t^2} \qquad (44.7)$$

Equating eqns. (0.6), (0.7) & (0.5),

$$\mathrm{FP} \times \mathrm{FJ} = \mathrm{AQ} \times \mathrm{F}\lambda = \mathrm{FN}^2 \qquad (44.8)$$

Eqn. (44.8) is mathematical expression of the theorem.

THEOREM- 45:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{JN \times PS = 2 \times FN^2 = 2 \times FZ^2}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (B.8)} \Longrightarrow \mathrm{JN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \qquad (45.1)$$

$$\text{Eqn.}\,(\text{A}.14) \Longrightarrow \text{PS} = 2\text{a}\sqrt{\text{t}^2+1} \quad (45.2)$$

Multiplying eqns. (45.1) & (45.2)

$$\text{JN} \times \text{PS} = \left(\frac{\text{a}(\text{t}^2+1)\sqrt{\text{t}^2+1}}{\text{t}^2}\right) \times 2\text{a}\sqrt{\text{t}^2+1}$$

$$\therefore \text{JN} \times \text{PS} = \frac{2\text{a}^2(\text{t}^2+1)(\text{t}^2+1)}{\text{t}^2}$$

$$\therefore \text{JN} \times \text{PS} = \frac{2\text{a}^2(\text{t}^2+1)^2}{\text{t}^2} \quad (45.3)$$

Referring fig. 1,

$$\text{Eqn.}\,(\text{B}.2) \Longrightarrow \text{FN} = \frac{\text{a}(\text{t}^2+1)}{\text{t}}$$

$$\therefore \text{FN}^2 = \frac{\text{a}^2(\text{t}^2+1)^2}{\text{t}^2}$$

Multiplying above eqn. by 2

$$\therefore 2 \times \text{FN}^2 = \frac{2\text{a}^2(\text{t}^2+1)^2}{\text{t}^2} \quad (45.4)$$

Referring fig. 1,

$$\text{Eqn.}\,(\text{B}.10) \Longrightarrow \text{FZ} = \frac{\text{a}(\text{t}^2+1)}{\text{t}}$$

$$\text{FZ} = \frac{\text{a}^2(\text{t}^2+1)^2}{\text{t}^2}$$

Multiplying above eqn. by 2

$$\therefore 2 \times \text{FZ}^2 = \frac{2\text{a}^2(\text{t}^2+1)^2}{\text{t}^2} \quad (45.5)$$

Equating eqns. (45.3), (45.4) & (45.5),

$$\text{JN} \times \text{PS} = 2 \times \text{FN}^2 = 2 \times \text{FZ}^2 \quad (45.6)$$

Eqn. (45.6) is mathematical expression of the theorem.

THEOREM- 46:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{OA \times OW = O\lambda}$

Derivations for proof of the theorem

Referring fig. 6,

$$\text{Eqn.}\,(\text{D}.11) \Longrightarrow \text{OA} = \text{a} \tag{46.1}$$

$$\text{Eqn.}\,(\text{D}.11) \Longrightarrow \text{OW} = \frac{1}{t^2} \tag{46.2}$$

$$\text{Eqn.}\,(\text{E}.14) \Longrightarrow \text{O}\lambda = \frac{a}{t^2} \tag{46.3}$$

Multiplying eqns. (46.1) & (46.2),

$$\therefore \text{OA} \times \text{OW} = a \times \frac{1}{t^2}$$

$$\therefore \text{OA} \times \text{OW} = \frac{a}{t^2} \tag{46.4}$$

Equating eqns. (46.3), (46.4)

$$\text{OA} \times \text{OW} = \text{O}\lambda \tag{46.3}$$

Eqn. (46.3) is mathematical expression of the theorem.

THEOREM- 47:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{OT \times OS = OB \times FC}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn.}\,(\text{A}.11) \Longrightarrow \text{OT} = at^2 \tag{47.1}$$

$$\text{Eqn. (A. 10)} \Longrightarrow \text{OS} = \text{a}(\text{t}^2 + 2) \qquad (47.2)$$

$$\text{Eqn. (C. 1)} \Longrightarrow \text{OB} = \frac{\text{at}(\text{t}^2 + 2)}{\sqrt{\text{t}^2 + 1}} \qquad (47.3)$$

$$\text{Eqn. (C. 2)} \Longrightarrow \text{FC} = \text{at}\sqrt{\text{t}^2 + 1} \qquad (47.4)$$

Multiplying eqns. (47.1) & (47.2),

$$\therefore \text{OT} \times \text{OS} = \text{at}^2 \times \text{a}(\text{t}^2 + 2)$$

$$\therefore \text{OT} \times \text{OS} = \text{a}^2\text{t}^2(\text{t}^2 + 2) \qquad (47.5)$$

Multiplying eqns. (47.3) & (47.4)

$$\text{OB} \times \text{FC} = \left(\frac{\text{at}(\text{t}^2 + 2)}{\sqrt{\text{t}^2 + 1}}\right) \times \text{at}\sqrt{\text{t}^2 + 1}$$

$$\therefore \text{OB} \times \text{FC} = \text{a}^2\text{t}^2(\text{t}^2 + 2) \qquad (47.6)$$

Equating the eqns. (47.5) & (47.6)

$$\text{OT} \times \text{OS} = \text{OB} \times \text{FC} \qquad (47.7)$$

Eqn. (47.7) is mathematical expression of the theorem.

THEOREM- 48:

In a parabola $y^2 = 4ax$, the parametric equation of parabola is $(x, y) = (at^2, 2at)$,

$\boldsymbol{PT \times PS = PQ \times TS}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 13)} \Longrightarrow \text{PT} = 2\text{at}\sqrt{\text{t}^2 + 1} \qquad (48.1)$$

$$\text{Eqn. (A. 14)} \Longrightarrow \text{PS} = 2\text{a}\sqrt{\text{t}^2 + 1} \qquad (48.2)$$

$$\text{Eqn.}(A.2) \Rightarrow PQ = 2at \quad (48.3)$$

$$\text{Eqn.}(A.15) \Rightarrow TS = 2a(t^2 + 1) \quad (48.4)$$

Multiplying eqns. (48.1) & (48.2),

$$\therefore PT \times PS = 2at\sqrt{t^2+1} \times 2a\sqrt{t^2+1}$$

$$\therefore PT \times PS = 4a^2t(t^2+1) \quad (48.5)$$

Multiplying eqns. (48.3) & (48.4)

$$PQ \times TS = 2at \times 2a(t^2+1)$$

$$\therefore PQ \times TS = 4a^2t(t^2+1) \quad (48.6)$$

Equating the eqns. (48.5) & (48.6)

$$PT \times PS = PQ \times TS \quad (48.7)$$

Eqn. (48.7) is mathematical expression of the theorem.

THEOREM- 49:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FC \times QD = 2 \times OV \times FM}$

Derivations for proof of the theorem

Referring fig. 2,

$$\text{Eqn.}(B.7) \Rightarrow OV = \frac{at^2}{\sqrt{t^2+1}} \quad (49.1)$$

$$\text{Eqn.}(A.18) \Rightarrow FM = a\sqrt{t^2+1} \quad (49.2)$$

$$\text{Eqn.}(C.2) \Rightarrow FC = at\sqrt{t^2+1} \quad (49.3)$$

$$\text{Eqn.}(C.3) \Longrightarrow QD = \frac{2at}{\sqrt{t^2+1}} \quad (49.4)$$

$$\text{Eqn.}(A.2) \Longrightarrow PQ = 2at$$

$$\therefore PQ^2 = 4a^2t^2$$

$$\therefore \frac{PQ^2}{2} = 2a^2t^2 \quad (49.5)$$

Multiplying eqns. (49.1) & (49.2),

$$OV \times FM = \frac{at^2}{\sqrt{t^2+1}} \times a\sqrt{t^2+1}$$

$$\therefore OV \times FM = a^2t^2$$

Multiplying above eqn. by 2,

$$\therefore 2 \times OV \times FM = 2a^2t^2 \quad (49.6)$$

Multiplying eqns. (49.3) & (49.4),

$$FC \times QD = at\sqrt{t^2+1} \times \frac{2at}{\sqrt{t^2+1}}$$

$$\therefore FC \times QD = 2a^2t^2 \quad (49.7)$$

Equating the eqns. (49.5) & (49.6)

$$FC \times QD = 2(OV \times FM) = \frac{PQ^2}{2} \quad (49.8)$$

Eqn. (49.8) is mathematical expression of the theorem.

THEOREM- 50:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{NM} \times \boldsymbol{VM} = \mathbf{OA^2} = \mathbf{OF^2} = \boldsymbol{a^2}$

Derivations for proof of the theorem

Referring fig. 2,

$$\text{Eqn. (B. 11)} \Longrightarrow \text{NM} = \frac{\text{a}\sqrt{\text{t}^2+1}}{\text{t}} \qquad (50.1)$$

$$\text{Eqn. (D. 23)} \Longrightarrow \text{VM} = \frac{\text{at}}{\sqrt{\text{t}^2+1}} \qquad (50.2)$$

$$\text{Eqn. (A. 4)} \Longrightarrow \text{OA} = \text{a}$$

$$\therefore \text{OA}^2 = \text{a}^2 \qquad (50.3)$$

$$\text{Eqn. (A. 4)} \Longrightarrow \text{OF} = \text{a}$$

$$\therefore \text{OF}^2 = \text{a}^2 \qquad (50.4)$$

Multiplying eqns. (50.1) & (50.2),

$$\text{NM} \times \text{VM} = \left(\frac{\text{a}\sqrt{\text{t}^2+1}}{\text{t}}\right) \times \left(\frac{\text{at}}{\sqrt{\text{t}^2+1}}\right)$$

$$\therefore \text{NM} \times \text{VM} = \left(\frac{\text{a}\sqrt{\text{t}^2+1}}{\text{t}}\right) \times \left(\frac{\text{at}}{\sqrt{\text{t}^2+1}}\right)$$

$$\therefore \text{NM} \times \text{VM} = \text{a}^2 \qquad (50.5)$$

Equating eqns. (50.5), (50.3) & (50.4),

$$\text{NM} \times \text{VM} = \text{OA}^2 = \text{OF}^2 = \text{a}^2$$

Eqn. (50.5) is mathematical expression of the theorem.

THEOREM- 51:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{TW \times T\lambda = QW \times Q\lambda}$$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (D. 13)} \Longrightarrow \text{TW} = \frac{\text{a}(\text{t}^4+1)}{\text{t}^2} \qquad (51.1)$$

$$\text{Eqn. (E.12)} \Longrightarrow \mathrm{T\lambda} = \frac{a(t^4-1)}{t^2} \qquad (51.2)$$

$$\text{Eqn. (D.23)} \Longrightarrow \mathrm{QW} = \frac{a(t^4-1)}{t^2} \qquad (51.3)$$

$$\text{Eqn. (E.17)} \Longrightarrow \mathrm{Q\lambda} = \frac{a(t^4+1)}{t^2} \qquad (51.4)$$

Multiplying eqns. (51.1) by (51.2),

$$\mathrm{TW} \times \mathrm{T\lambda} = \left(\frac{a(t^4+1)}{t^2}\right) \times \left(\frac{a(t^4-1)}{t^2}\right)$$

$$\therefore \mathrm{TW} \times \mathrm{T\lambda} = \frac{a^2(t^8-1)}{t^4} \qquad (51.5)$$

Dividing eqns. (51.3) by (51.4),

$$\mathrm{QW} \times \mathrm{Q\lambda} = \left(\frac{a(t^4+1)}{t^2}\right) \times \left(\frac{a(t^4-1)}{t^2}\right)$$

$$\therefore \mathrm{QW} \times \mathrm{Q\lambda} = \frac{a^2(t^8-1)}{t^4} \qquad (51.6)$$

Equating the above eqn. (51.5) & (51.6)

$$\mathrm{TW} \times \mathrm{T\lambda} = \mathrm{QW} \times \mathrm{Q\lambda} \qquad (51.7)$$

Eqn. (51.7) is mathematical expression of the theorem.

THEOREM- 52:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{JW \times PQ = \mathrm{OT} \times \mathrm{OW} = 2 \times \mathrm{OF}^2 = 2a^2}$$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn.}(A.5) \Longrightarrow JW = \frac{2a}{t} \tag{52.1}$$

$$\text{Eqn.}(A.19) \Longrightarrow PQ = 2at \tag{52.2}$$

$$\text{Eqn.}(A.12) \Longrightarrow TQ = 2at^2 \tag{52.3}$$

$$\text{Eqn.}(E.15) \Longrightarrow W\lambda = \frac{2a}{t^2} \tag{52.4}$$

$$\text{Eqn.}(A.6) \Longrightarrow SQ = 2a$$

$$\therefore SQ^2 = 4a^2 \tag{52.5}$$

$$\text{Eqn.}(A.19) \Longrightarrow OF = a$$

$$\therefore 4 \times OF^2 = 4a^2 \tag{52.6}$$

$$\text{Eqn.}(A.5) \Longrightarrow FG = FH = 2a$$

$$\therefore FG^2 = FH^2 = 4a^2 \tag{52.7}$$

Multiplying eqns. (52.1) & (52.2),

$$JW \times PQ = \frac{2a}{t} \times 2at$$

$$\therefore JW \times PQ = 4a^2 \tag{52.8}$$

Multiplying eqns. (52.3) & (52.4),

$$TQ \times W\lambda = 2at^2 \times \frac{2a}{t^2}$$

$$\therefore TQ \times W\lambda = 4a^2 \tag{52.9}$$

Equating eqns. (52.4), (52.5), (52.7) & (52.3),

$$JW \times PQ = TQ \times W\lambda = SQ^2 = 4 \times OF^2 = 2 \times FG^2 = 2 \times FH^2 \tag{52.10}$$

Eqn. (52.10) is mathematical expression of the theorem.

THEOREM- 53:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\mathrm{FM \times S\Omega = FT \times FW = TA \times AW = AN \times FN = AL \times JN = TL \times NM}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (A.18)} \Longrightarrow \mathrm{FM} = a\sqrt{t^2+1} \tag{53.1}$$

$$\text{Eqn. (E.11)} \Longrightarrow \mathrm{S\Omega} = \frac{a(t^2-1)\sqrt{t^2+1}}{t^2} \tag{53.2}$$

$$\text{Eqn. (A.16)} \Longrightarrow \mathrm{FT} = a(t^2+1) \tag{53.3}$$

$$\text{Eqn. (D.10)} \Longrightarrow \mathrm{FW} = \frac{a(t^2-1)}{t^2} \tag{53.4}$$

$$\text{Eqn. (A.21)} \Longrightarrow \mathrm{TA} = a(t^2-1) \tag{53.5}$$

$$\text{Eqn. (D.22)} \Longrightarrow \mathrm{AW} = \frac{a(t^2+1)}{t^2} \tag{53.6}$$

$$\text{Eqn. (B.1)} \Longrightarrow \mathrm{AN} = \frac{a(t^2-1)}{t} \tag{53.7}$$

$$\text{Eqn. (B.2)} \Longrightarrow \mathrm{FN} = \frac{a(t^2+1)}{t} \tag{53.8}$$

$$\text{Eqn. (A.6)} \Longrightarrow \mathrm{AL} = \frac{a(t^2-1)}{\sqrt{t^2+1}} \tag{53.9}$$

$$\text{Eqn. (A.6)} \Longrightarrow \text{JN} = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \tag{53.10}$$

$$\text{Eqn. (E.26)} \Longrightarrow \text{TL} = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \tag{53.11}$$

$$\text{Eqn. (B.11)} \Longrightarrow \text{NM} = \frac{a\sqrt{t^2+1}}{t} \tag{53.12}$$

$$\text{Eqn. (C.9)} \Longrightarrow \text{SC} = a\sqrt{t^2+1} \tag{53.13}$$

$$\text{Eqn. (E.7)} \Longrightarrow \text{N}\mu = a\sqrt{t^2+1} \tag{53.14}$$

Multiplying eqns. (53.1) & (53.2),

$$\text{FM} \times \text{S}\Omega = a\sqrt{t^2+1} \times \frac{a(t^2-1)\sqrt{t^2+1}}{t^2}$$

$$\therefore \text{FM} \times \text{S}\Omega = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.15}$$

Multiplying eqns. (53.3) & (53.4),

$$\text{FT} \times \text{FW} = a(t^2+1) \times \frac{a(t^2-1)}{t^2}$$

$$\therefore \text{FT} \times \text{FW} = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.16}$$

Multiplying eqns. (53.3) & (53.4),

$$\text{TA} \times \text{AW} = a(t^2-1) \times \frac{a(t^2+1)}{t^2}$$

$$\therefore \text{TA} \times \text{AW} = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.17}$$

Multiplying eqns. (53.3) & (53.4),

$$\mathrm{AN} \times \mathrm{FN} = \frac{a(t^2-1)}{t} \times \frac{a(t^2+1)}{t}$$

$$\therefore \mathrm{AN} \times \mathrm{FN} = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.18}$$

Multiplying eqns. (53.3) & (53.4),

$$\mathrm{AL} \times \mathrm{JN} = \frac{a(t^2-1)}{\sqrt{t^2+1}} \times \frac{a(t^2+1)\sqrt{t^2+1}}{t^2}$$

$$\therefore \mathrm{AL} \times \mathrm{JN} = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.19}$$

Multiplying eqns. (53.3) & (53.4),

$$\mathrm{TL} \times \mathrm{NM} = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \times \frac{a\sqrt{t^2+1}}{t}$$

$$\therefore \mathrm{TL} \times \mathrm{NM} = \frac{a^2(t^2+1)(t^2-1)}{t^2} \tag{53.53}$$

Equating eqns. (53.15), (53.16), (53.17), (53.18), (53.19), & (53.20),

$$\mathrm{FM} \times \mathrm{S\Omega} = \mathrm{FT} \times \mathrm{FW} = \mathrm{TA} \times \mathrm{AW} = \mathrm{AN} \times \mathrm{FN} = \mathrm{AL} \times \mathrm{JN} = \mathrm{TL} \times \mathrm{NM} \tag{53.21}$$

Eqn. (53.21) is mathematical expression of the theorem.

THEOREM- 54:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{P\Omega - S\Omega = J\lambda = PS \times OW = 2 \times \mu\lambda = 2 \times J\mu}$$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn.}\,(\text{E}.19) \Longrightarrow \mathrm{J\lambda} = \frac{2a\sqrt{t^2+1}}{t^2} \tag{54.1}$$

$$\text{Eqn.}\,(\text{A}.14) \Longrightarrow \mathrm{PS} = 2a\sqrt{t^2+1} \tag{54.2}$$

$$\text{Eqn. (D. 11)} \implies \text{OW} = \frac{1}{t^2} \quad (54.3)$$

Referring fig. 6,

$$\text{Eqn. (E. 20)} \implies \mu\lambda = \frac{a\sqrt{t^2+1}}{t^2}$$

Multiplying above eqn. by 2

$$\therefore 2 \times \mu\lambda = \frac{2a\sqrt{t^2+1}}{t^2} \quad (54.4)$$

$$\text{Eqn. (E. 8)} \implies \text{J}\mu = \frac{a\sqrt{t^2+1}}{t^2}$$

Multiplying above eqn. by 2

$$\therefore 2 \times \text{J}\mu = \frac{2a\sqrt{t^2+1}}{t^2} \quad (54.5)$$

$$\text{Eqn. (E. 10)} \implies \text{P}\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \quad (54.6)$$

$$\text{Eqn. (E. 11)} \implies \text{S}\Omega = \frac{a(t^2-1)\sqrt{t^2+1}}{t^2} \quad (54.7)$$

Multiplying eqns. (54.2) & (54.3)

$$\text{PS} \times \text{OW} = 2a\sqrt{t^2+1} \times \frac{1}{t^2}$$

$$\therefore \text{PS} \times \text{OW} = \frac{2a\sqrt{t^2+1}}{t^2} \quad (54.8)$$

$$\text{P}\Omega - \text{S}\Omega = \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t^2}\right) - \left(\frac{a(t^2-1)\sqrt{t^2+1}}{t^2}\right)$$

$$\therefore \text{P}\Omega - \text{S}\Omega = \frac{a\sqrt{t^2+1}(t^2+1-t^2-1)}{t^2}$$

$$\therefore P\Omega - S\Omega = \frac{2a\sqrt{t^2+1}}{t^2} \tag{54.9}$$

Equating eqns. (54.1), (54.4), (54.5), (54.8) & (54.9)

$$P\Omega - S\Omega = J\lambda = PS \times OW = 2 \times \mu\lambda = 2 \times J\mu \tag{54.10}$$

Eqn. (54.10) is mathematical expression of the theorem.

THEOREM- 55:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FQ + FP = 2 \times OQ}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 7)} \Longrightarrow FQ = a(t^2 - 1) \tag{55.1}$$

$$\text{Eqn. (A. 9)} \Longrightarrow FP = a(t^2 + 1) \tag{55.2}$$

$\text{Eqn. (A. 1)} \Longrightarrow OQ = at^2$

$$\therefore 2 \times OQ = 2at^2 \tag{55.3}$$

Multiplying eqns. (55.1) & (55.2)

$FQ + FP = a(t^2 - 1) + a(t^2 + 1)$

$\therefore FQ + FP = at^2 - a + at^2 + a$

$$\therefore FQ + FP = 2at^2 \tag{55.4}$$

Equating the eqns. (55.4) & (55.3),

$$FQ + FP = 2 \times OQ \tag{55.5}$$

Eqn. (55.5) is mathematical expression of the theorem.

THEOREM- 56:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{J\Omega - SZ = QM - SZ = NM}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (A.22)} \Longrightarrow \text{QM} = \text{at}\sqrt{\text{t}^2 + 1} \quad (56.1)$$

$$\text{Eqn. (E.9)} \Longrightarrow \text{J}\Omega = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}} \quad (56.2)$$

$$\text{Eqn. (D.17)} \Longrightarrow \text{SZ} = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}} \quad (56.3)$$

$$\text{Eqn. (B.11)} \Longrightarrow \text{NM} = \frac{\text{a}\sqrt{\text{t}^2 + 1}}{\text{t}} \quad (56.4)$$

Subtracting eqns. (56.2) from (56.1)

$$\text{J}\Omega - \text{QM} = \left(\frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}}\right) - \text{at}\sqrt{\text{t}^2 + 1}$$

$$\therefore \text{J}\Omega - \text{QM} = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1} - \text{at}^2\sqrt{\text{t}^2 + 1}}{\text{t}}$$

$$\therefore \text{J}\Omega - \text{QM} = \frac{\text{a}\sqrt{\text{t}^2 + 1}(\text{t}^2 + 1 - \text{t}^2)}{\text{t}}$$

$$\therefore \text{J}\Omega - \text{QM} = \frac{\text{a}\sqrt{\text{t}^2 + 1}}{\text{t}} \quad (56.5)$$

Subtracting eqns. (56.3) from (56.1)

$$\text{QM} - \text{SZ} = \left(\frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1}}{\text{t}}\right) - \text{at}\sqrt{\text{t}^2 + 1}$$

$$\therefore \text{QM} - \text{SZ} = \frac{\text{a}(\text{t}^2 + 1)\sqrt{\text{t}^2 + 1} - \text{at}^2\sqrt{\text{t}^2 + 1}}{\text{t}}$$

$$\therefore \mathrm{QM} - \mathrm{SZ} = \frac{a\sqrt{t^2+1}(t^2+1-t^2)}{t}$$

$$\therefore \mathrm{QM} - \mathrm{SZ} = \frac{a\sqrt{t^2+1}}{t} \qquad (56.6)$$

Equating the eqns. (56.5), (56.6) & (56.4),

$$\mathrm{J\Omega} - \mathrm{QM} = \mathrm{QM} - \mathrm{SZ} = \mathrm{NM} \qquad (56.7)$$

Eqn. (56.7) is mathematical expression of the theorem.

THEOREM- 57:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FJ + FW = 2a}$

Derivations for proof of the theorem

Referring fig. 6,

$$\text{Eqn.}\,(D.5) \Longrightarrow \mathrm{FJ} = \frac{a(t^2+1)}{t^2} \qquad (57.1)$$

$$\text{Eqn.}\,(D.10) \Longrightarrow \mathrm{FW} = \frac{a(t^2-1)}{t^2} \qquad (57.2)$$

Adding eqns. (57.1) & (57.2),

$$\mathrm{FJ} + \mathrm{FW} = \left(\frac{a(t^2+1)}{t^2}\right) + \left(\frac{a(t^2-1)}{t^2}\right)$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = \frac{at^2 + a + at^2 - a}{t^2}$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = \frac{at^2 + a + at^2 - a}{t^2}$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = \frac{2at^2}{t^2}$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = 2a \qquad (57.3)$$

Eqn. (57.3) is mathematical expression of the theorem.

THEOREM- 58:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{FM - S\Omega = JN - SC = JN - FM = JN - NM = J\mu = \mu\lambda}$$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (E. 11)} \Longrightarrow S\Omega = \frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t^2} \qquad (58.1)$$

$$\text{Eqn. (A. 18)} \Longrightarrow FM = a\sqrt{t^2 + 1} \qquad (58.2)$$

$$\text{Eqn. (B. 8)} \Longrightarrow JN = \frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t^2} \qquad (58.3)$$

$$\text{Eqn. (A. 20)} \Longrightarrow SC = a\sqrt{t^2 + 1} \qquad (58.4)$$

$$\text{Eqn. (A. 18)} \Longrightarrow FM = a\sqrt{t^2 + 1} \qquad (58.5)$$

$$\text{Eqn. (E. 7)} \Longrightarrow N\mu = a\sqrt{t^2 + 1} \qquad (58.6)$$

$$\text{Eqn. (E. 8)} \Longrightarrow J\mu = \frac{a\sqrt{t^2 + 1}}{t^2} \qquad (58.7)$$

$$\text{Eqn. (E. 20)} \Longrightarrow \mu\lambda = \frac{a\sqrt{t^2 + 1}}{t^2} \qquad (58.8)$$

Subtracting eqns. (58.2) & (58.1),

$$FM - S\Omega = a\sqrt{t^2 + 1} - \left(\frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t^2}\right)$$

$$\therefore FM - S\Omega = \frac{at^2\sqrt{t^2 + 1} - a(t^2 - 1)\sqrt{t^2 + 1}}{t^2}$$

$$\therefore \mathrm{FM} - \mathrm{S}\Omega = \frac{\mathrm{a}\sqrt{\mathrm{t}^2+1}(\mathrm{t}^2 - \mathrm{t}^2 + 1)}{\mathrm{t}^2}$$

$$\therefore \mathrm{FM} - \mathrm{S}\Omega = \frac{\mathrm{a}\sqrt{\mathrm{t}^2+1}}{\mathrm{t}^2} \tag{58.9}$$

Subtracting eqns. (58.2) & (58.1),

$$\mathrm{JN} - \mathrm{SC} = \left(\frac{\mathrm{a}(\mathrm{t}^2+1)\sqrt{\mathrm{t}^2+1}}{\mathrm{t}^2}\right) - \mathrm{a}\sqrt{\mathrm{t}^2+1}$$

$$\therefore \mathrm{JN} - \mathrm{SC} = \frac{\mathrm{a}(\mathrm{t}^2+1)\sqrt{\mathrm{t}^2+1} - \mathrm{at}^2\sqrt{\mathrm{t}^2+1}}{\mathrm{t}^2}$$

$$\therefore \mathrm{JN} - \mathrm{SC} = \frac{\mathrm{a}\sqrt{\mathrm{t}^2+1}(\mathrm{t}^2 + 1 - \mathrm{t}^2)}{\mathrm{t}^2}$$

$$\therefore \mathrm{JN} - \mathrm{SC} = \frac{\mathrm{a}\sqrt{\mathrm{t}^2+1}}{\mathrm{t}^2} \tag{58.10}$$

Equating the above eqn. (58.9), (58.10), (58.7), (58.8),

$$\mathrm{FM} - \mathrm{S}\Omega = \mathrm{JN} - \mathrm{SC} = \mathrm{JN} - \mathrm{FM} = \mathrm{JN} - \mathrm{NM} = \mathrm{J}\mu = \mu\lambda \tag{58.11}$$

Eqn. (58.3) is mathematical expression of the theorem.

THEOREM- 59:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{OS - TA = 3a}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A.10)} \Longrightarrow \mathrm{OS} = \mathrm{a}(\mathrm{t}^2 + 2) \tag{59.1}$$

$$\text{Eqn. (A.21)} \Longrightarrow \mathrm{TA} = \mathrm{a}(\mathrm{t}^2 - 1) \tag{59.2}$$

Subtracting eqns. (59.2) from (59.1),

$$\mathrm{OS} - \mathrm{TA} = \mathrm{a(t^2 + 2) - a(t^2 - 1)}$$

$$\therefore \mathrm{OS} - \mathrm{TA} = \mathrm{a(t^2 + 2) - a(t^2 - 1)}$$

$$\therefore \mathrm{OS} - \mathrm{TA} = \mathrm{(at^2 + 2a) - (at^2 - a)}$$

$$\therefore \mathrm{OS} - \mathrm{TA} = \mathrm{at^2 + 2a - at^2 + a}$$

$$\therefore \mathrm{OS} - \mathrm{TA} = 3\mathrm{a} \qquad (59.3)$$

Eqn. (59.3) is mathematical expression of the theorem.

THEOREM- 60:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$$\boldsymbol{AE - OB = \frac{QD}{2}}$$

Derivations for proof of the theorem

Referring fig. 3,

$$\mathrm{Eqn.\,(C.11)} \Longrightarrow \mathrm{AE} = \frac{\mathrm{at(t^2 + 3)}}{\sqrt{\mathrm{t^2 + 1}}} \qquad (60.1)$$

$$\mathrm{Eqn.\,(C.1)} \Longrightarrow \mathrm{OB} = \frac{\mathrm{at(t^2 + 2)}}{\sqrt{\mathrm{t^2 + 1}}} \qquad (60.2)$$

$$\mathrm{Eqn.\,(C.3)} \Longrightarrow \mathrm{QD} = \frac{2\mathrm{at}}{\sqrt{\mathrm{t^2 + 1}}} \qquad (60.3)$$

Subtracting eqns. (60.2) from (60.1),

$$\mathrm{AE} - \mathrm{OB} = \left(\frac{\mathrm{at(t^2 + 3)}}{\sqrt{\mathrm{t^2 + 1}}}\right) - \left(\frac{\mathrm{at(t^2 + 2)}}{\sqrt{\mathrm{t^2 + 1}}}\right)$$

$$\therefore \mathrm{AE} - \mathrm{OB} = \frac{\mathrm{at(t^2 + 3) - at(t^2 + 2)}}{\sqrt{\mathrm{t^2 + 1}}}$$

$$\therefore \mathrm{AE} - \mathrm{OB} = \frac{\mathrm{at[t^2 + 3 - t^2 - 2]}}{\sqrt{\mathrm{t^2 + 1}}}$$

$$\therefore AE - OB = \frac{at}{\sqrt{t^2+1}} \quad (60.4)$$

Dividing eqn. (60.3) by 2,

$$\frac{QD}{2} = \frac{2at}{\sqrt{t^2+1}} \times \frac{1}{2}$$

$$\therefore \frac{QD}{2} = \frac{2at}{\sqrt{t^2+1}} \times \frac{1}{2}$$

$$\therefore \frac{QD}{2} = \frac{at}{\sqrt{t^2+1}} \quad (60.5)$$

Equating the above eqn. (60.4) & (60.5)

$$AE - OB = \frac{QD}{2} \quad (60.6)$$

Eqn. (60.6) is mathematical expression of the theorem.

THEOREM- 61:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{TW + T\lambda = Q\lambda + T\lambda = 2 \times OQ}$$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (D. 13)} \Rightarrow TW = \frac{a(t^4+1)}{t^2} \quad (61.1)$$

$$\text{Eqn. (E. 12)} \Rightarrow T\lambda = \frac{a(t^4-1)}{t^2} \quad (61.2)$$

$$\text{Eqn. (D. 13)} \Rightarrow Q\lambda = \frac{a(t^4+1)}{t^2} \quad (61.3)$$

$$\text{Eqn. (E.12)} \Longrightarrow T\lambda = \frac{a(t^4 - 1)}{t^2} \quad (61.4)$$

$$\text{Eqn. (A.1)} \Longrightarrow OQ = at^2$$

Multiplying above eqn. by 2,

$$\therefore 2 \times OQ = 2at^2 \quad (61.5)$$

Adding eqns. (61.1) & (61.2),

$$\therefore TW + T\lambda = \left(\frac{a(t^4 - 1)}{t^2}\right) + \left(\frac{a(t^4 + 1)}{t^2}\right)$$

$$\therefore TW + T\lambda = \frac{a(t^4 - 1) + a(t^4 + 1)}{t^2}$$

$$\therefore TW + T\lambda = \frac{a(t^4 - 1 + t^4 + 1)}{t^2}$$

$$\therefore TW + T\lambda = \frac{2at^4}{t^2}$$

$$\therefore TW + T\lambda = 2at^2 \quad (61.6)$$

Adding eqns. (61.3) & (61.4),

$$Q\lambda + T\lambda = \left(\frac{a(t^4 - 1)}{t^2}\right) + \left(\frac{a(t^4 + 1)}{t^2}\right)$$

$$\therefore Q\lambda + T\lambda = \frac{a(t^4 - 1) + a(t^4 + 1)}{t^2}$$

$$\therefore Q\lambda + T\lambda = \frac{a(t^4 - 1 + t^4 + 1)}{t^2}$$

$$\therefore Q\lambda + T\lambda = \frac{2at^4}{t^2}$$

$$\therefore Q\lambda + T\lambda = 2at^2 \quad (61.7)$$

Equating the above eqn. (61.6) (61.7) & (61.5)

$$TW + T\lambda = T\lambda + Q\lambda = 2 \times OQ \qquad (61.8)$$

Eqn. (00.5) is mathematical expression of the theorem.

THEOREM- 62:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{P\Omega + S\Omega = FM}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (E. 10)} \Longrightarrow P\Omega = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \qquad (62.1)$$

$$\text{Eqn. (E. 11)} \Longrightarrow S\Omega = \frac{a(t^2-1)\sqrt{t^2+1}}{t^2} \qquad (62.2)$$

$\text{Eqn. (A. 18)} \Longrightarrow FM = a\sqrt{t^2+1}$

Multiplying above eqn. by 2

$$\therefore 2 \times FM = 2a\sqrt{t^2+1} \qquad (62.3)$$

Adding eqns. (62.1) & (62.2),

$$\therefore P\Omega + S\Omega = \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t^2}\right) + \left(\frac{a(t^2-1)\sqrt{t^2+1}}{t^2}\right)$$

$$\therefore P\Omega + S\Omega = \frac{a(t^2+1)\sqrt{t^2+1} + a(t^2-1)\sqrt{t^2+1}}{t^2}$$

$$\therefore P\Omega + S\Omega = \frac{a\sqrt{t^2+1}(t^2+1+t^2-1)}{t^2}$$

$$\therefore P\Omega + S\Omega = \frac{a\sqrt{t^2+1}(t^2+1+t^2-1)}{t^2}$$

$$\therefore P\Omega + S\Omega = \frac{2a\sqrt{t^2+1}(t^2)}{t^2}$$

$$\therefore P\Omega + S\Omega = 2a\sqrt{t^2 + 1} \quad (62.4)$$

Equating the above eqn. (62.3) & (62.4)

$$P\Omega + S\Omega = 2FM \quad (62.5)$$

Eqn. (62.5) is mathematical expression of the theorem.

THEOREM- 63:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TW - QW = AF \times OW}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn.}\ (D.13) \Longrightarrow TW = \frac{a(t^4 + 1)}{t^2} \quad (63.1)$$

$$\text{Eqn.}\ (D.23) \Longrightarrow QW = \frac{a(t^4 - 1)}{t^2} \quad (63.2)$$

$$\text{Eqn.}\ (A.35) \Longrightarrow AF = 2a \quad (63.3)$$

$$\text{Eqn.}\ (D.11) \Longrightarrow OW = \frac{1}{t^2} \quad (63.4)$$

Subtracting eqns. (63.2) from (63.1),

$$\therefore TW - QW = \left(\frac{a(t^4 + 1)}{t^2}\right) - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore TW - QW = \frac{a(t^4 + 1) - a(t^4 - 1)}{t^2}$$

$$\therefore TW - QW = \frac{a(t^4 + 1 - t^4 + 1)}{t^2}$$

$$\therefore TW - QW = \frac{2a}{t^2} \quad (63.5)$$

Multiplying eqns. (63.3) & (63.4)

$$\mathrm{AF} \times \mathrm{OW} = 2\mathrm{a} \times \frac{1}{\mathrm{t}^2}$$

Equating the above eqn. (63.4) & (63.5)

$$\mathrm{TW} - \mathrm{QW} = \mathrm{W}\lambda \qquad (63.5)$$

Eqn. (63.5) is mathematical expression of the theorem.

THEOREM- 64:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{TW + QW = 2 \times OQ}$$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (D. 13)} \Longrightarrow \mathrm{TW} = \frac{\mathrm{a}(\mathrm{t}^4 + 1)}{\mathrm{t}^2} \qquad (64.1)$$

$$\text{Eqn. (D. 23)} \Longrightarrow \mathrm{QW} = \frac{\mathrm{a}(\mathrm{t}^4 - 1)}{\mathrm{t}^2} \qquad (64.2)$$

$$\text{Eqn. (A. 1)} \Longrightarrow \mathrm{OQ} = \mathrm{at}^2$$

Multiplying above eqn. by 2

$$2 \times \mathrm{OQ} = 2\mathrm{at}^2 \qquad (64.3)$$

Adding eqns. (64.2) from (64.1),

$$\therefore \mathrm{TW} + \mathrm{QW} = \left(\frac{\mathrm{a}(\mathrm{t}^4 + 1)}{\mathrm{t}^2}\right) + \left(\frac{\mathrm{a}(\mathrm{t}^4 - 1)}{\mathrm{t}^2}\right)$$

$$\therefore \mathrm{TW} + \mathrm{QW} = \frac{\mathrm{a}(\mathrm{t}^4 + 1) + \mathrm{a}(\mathrm{t}^4 - 1)}{\mathrm{t}^2}$$

$$\therefore \mathrm{TW} + \mathrm{QW} = \frac{\mathrm{a}(\mathrm{t}^4 + 1 + \mathrm{t}^4 - 1)}{\mathrm{t}^2}$$

$$\therefore \mathrm{TW} + \mathrm{QW} = \frac{2\mathrm{at}^4}{\mathrm{t}^2}$$

$$\therefore \mathrm{TW} + \mathrm{QW} = \frac{2at^4}{t^2}$$

$$\therefore \mathrm{TW} + \mathrm{QW} = 2at^2 \qquad (64.4)$$

Equating the above eqn. (64.4) & (64.3)

$$\mathrm{TW} + \mathrm{QW} = 2 \times 0\mathrm{Q} \qquad (64.5)$$

Eqn. (64.5) is mathematical expression of the theorem.

THEOREM- 65:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TW - T\lambda = W\lambda}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (E. 12)} \Rightarrow \mathrm{T}\lambda = \frac{a(t^4 - 1)}{t^2} \qquad (65.1)$$

$$\text{Eqn. (D. 13)} \Rightarrow \mathrm{TW} = \frac{a(t^4 + 1)}{t^2} \qquad (65.2)$$

$$\text{Eqn. (E. 15)} \Rightarrow \mathrm{W}\lambda = \frac{2a}{t^2} \qquad (65.3)$$

Subtracting eqns. (65.2) from (65.1),

$$\therefore \mathrm{TW} - \mathrm{T}\lambda = \left(\frac{a(t^4 + 1)}{t^2}\right) - \left(\frac{a(t^4 - 1)}{t^2}\right)$$

$$\therefore \mathrm{TW} - \mathrm{T}\lambda = \frac{a(t^4 + 1) - a(t^4 - 1)}{t^2}$$

$$\therefore \mathrm{TW} - \mathrm{T}\lambda = \frac{a(t^4 + 1 - t^4 + 1)}{t^2}$$

$$\therefore \mathrm{TW} - \mathrm{T}\lambda = \frac{2a}{t^2} \qquad (65.4)$$

Equating the above eqn. (65.3) & (65.4)

$$TW - T\lambda = W\lambda \quad (65.5)$$

Eqn. (65.5) is mathematical expression of the theorem.

THEOREM- 66:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FQ + FP = 2 \times OQ}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 7)} \Longrightarrow FQ = a(t^2 - 1) \quad (66.1)$$

$$\text{Eqn. (A. 9)} \Longrightarrow FP = a(t^2 + 1) \quad (66.2)$$

Adding eqns. (1.1) & (1.2),

$$FQ + FP = at^2 - 1 + at^2 + 1$$

$$FQ + FP = 2at^2 \quad (66.3)$$

$$\text{Eqn. (A. 1)} \Longrightarrow OQ = at^2$$

$$\therefore 2 \times OQ = 2at^2 \quad (66.4)$$

Equating eqns. (0.3) & (0.4),

$$FQ + FP = 2 \times OQ \quad (66.5)$$

Eqn. (66.5) is mathematical expression of the theorem.

THEOREM- 67:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FP - FQ = 2a}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A.9)} \Longrightarrow \text{FP} = \text{a}(\text{t}^2 + 1) \tag{67.1}$$

$$\text{Eqn. (A.7)} \Longrightarrow \text{FQ} = \text{a}(\text{t}^2 - 1) \tag{67.2}$$

Subtracting eqn. (67.2) from (67.1)

$$\text{FP} - \text{FQ} = \text{a}(\text{t}^2 + 1) - \text{a}(\text{t}^2 - 1)$$

$$\therefore \text{FP} - \text{FQ} = \text{at}^2 + \text{a} - \text{at}^2 + \text{a}$$

$$\therefore \text{FP} - \text{FQ} = 2\text{a} \tag{67.3}$$

Eqn. (67.3) is mathematical expression of the theorem.

THEOREM- 68:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FJ + FW = 2a}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (D.5)} \Longrightarrow \text{FJ} = \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2} \tag{68.1}$$

$$\text{Eqn. (D.10)} \Longrightarrow \text{FW} = \frac{\text{a}(\text{t}^2 - 1)}{\text{t}^2} \tag{68.2}$$

Adding eqns. (0.1) & (0.2),

$$\text{FJ} + \text{FW} = \frac{\text{a}(\text{t}^2 + 1)}{\text{t}^2} + \frac{\text{a}(\text{t}^2 - 1)}{\text{t}^2}$$

$$\therefore \text{FJ} + \text{FW} = \frac{\text{at}^2 + \text{a} + \text{at}^2 - \text{a}}{\text{t}^2}$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = \frac{2at^2}{t^2}$$

$$\therefore \mathrm{FJ} + \mathrm{FW} = 2a \qquad (68.3)$$

Eqn. (68.3) is mathematical expression of the theorem.

THEOREM- 69:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{AN + FN = 2 \times OQ = 2 \times OT = TQ}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{AN} = a(t^2 - 1) \qquad (69.1)$$

$$\text{Eqn. (A.69)} \Longrightarrow \mathrm{FN} = a(t^2 + 1) \qquad (69.2)$$

Adding eqns. (69.1) & (69.2),

$$\mathrm{AN} + \mathrm{FN} = a(t^2 - 1) + a(t^2 + 1)$$

$$\therefore \mathrm{AN} + \mathrm{FN} = at^2 - a + at^2 + a$$

$$\therefore \mathrm{AN} + \mathrm{FN} = 2at^2 \qquad (69.3)$$

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{OQ} = at^2$$

$$\therefore 2 \times \mathrm{OQ} = 2at^2 \qquad (69.4)$$

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{OT} = at^2$$

$$\therefore 2 \times \mathrm{OT} = 2at^2 \qquad (69.5)$$

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{TQ} = 2at^2 \qquad (69.6)$$

Equating eqns. (69.3), (69.4), (69.5) & (69.6),

$\mathrm{AN} + \mathrm{FN} = 2 \times \mathrm{OQ} = 2 \times \mathrm{OT} = \mathrm{TQ}$

Eqn. (69.5) is mathematical expression of the theorem.

THEOREM- 70:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TN + SZ = 2 \times QM = 2 \times PM = 2 \times TM = 2 \times FC = PT}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (B.3)} \Longrightarrow \mathrm{TN} = \frac{\mathrm{a(t^2 - 1)\sqrt{t^2 + 1}}}{\mathrm{t}} \tag{70.1}$$

$$\text{Eqn. (D.21)} \Longrightarrow \mathrm{SZ} = \frac{\mathrm{a(t^2 + 1)\sqrt{t^2 + 1}}}{\mathrm{t}} \tag{70.2}$$

$$\text{Eqn. (A.13)} \Longrightarrow \mathrm{PT} = 2\mathrm{at}\sqrt{\mathrm{t^2 + 1}} \tag{70.3}$$

$$\text{Eqn. (A.22)} \Longrightarrow \mathrm{QM} = \mathrm{at}\sqrt{\mathrm{t^2 + 1}}$$

$$\therefore 2 \times \mathrm{QM} = 2\mathrm{at}\sqrt{\mathrm{t^2 + 1}} \tag{70.4}$$

$$\text{Eqn. (A.20)} \Longrightarrow \mathrm{PM} = \mathrm{at}\sqrt{\mathrm{t^2 + 1}}$$

$$\therefore 2 \times \mathrm{PM} = 2\mathrm{at}\sqrt{\mathrm{t^2 + 1}} \tag{70.5}$$

$$\text{Eqn. (A.19)} \Longrightarrow \mathrm{TM} = \mathrm{at}\sqrt{\mathrm{t^2 + 1}}$$

$$\therefore 2 \times \mathrm{TM} = 2\mathrm{at}\sqrt{\mathrm{t^2 + 1}} \tag{70.6}$$

$$\text{Eqn. (C.2)} \Longrightarrow \mathrm{FC} = \mathrm{at}\sqrt{\mathrm{t^2 + 1}}$$

$$\therefore 2 \times \mathrm{FC} = 2\mathrm{at}\sqrt{\mathrm{t^2 + 1}} \tag{70.7}$$

Adding eqns. (70.1) & (70.2),

$$TN + SZ = \frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t} + \frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}$$

$$\therefore TN + SZ = \frac{a(t^2 - 1)\sqrt{t^2 + 1} + a(t^2 + 1)\sqrt{t^2 + 1}}{t}$$

$$\therefore TN + SZ = \frac{a\sqrt{t^2 + 1}(t^2 - 1 + t^2 + 1)}{t}$$

$$\therefore TN + SZ = \frac{a\sqrt{t^2 + 1} \times (2t^2)}{t}$$

$$\therefore TN + SZ = 2at\sqrt{t^2 + 1} \quad (70.8)$$

Equating eqns. (70.8), (70.4), (70.5), (70.6), (70.7) & (70.3),

$$TN + SZ = 2 \times QM = 2 \times PM = 2 \times TM = 2 \times FC = PT \quad (70.9)$$

Eqn. (70.9) is mathematical expression of the theorem.

THEOREM- 71:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PQ^2 + QG^2 = SG^2}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn.}\ (A.26) \Longrightarrow SG = a\sqrt{(t^2 + 1)^2 + 4}$$

$$\therefore SG^2 = a^2[(t^2 + 1)^2 + 4] \quad (71.1)$$

$$\text{Eqn.}\ (A.24) \Longrightarrow QG = a\sqrt{(t^2 - 1)^2 + 4}$$

$$\therefore QG^2 = a^2[(t^2 - 1)^2 + 4] \quad (71.2)$$

$$\text{Eqn.}\ (A.2) \Longrightarrow PQ = 2at$$

$$\therefore PQ^2 = 4a^2t^2 \quad (71.3)$$

Subtracting eqns. (71.2) from (71.1),

$SG^2 - QG^2 = [a^2(t^2+1)^2 + 4a^2] - [a^2(t^2-1)^2 + 4a^2]$

$\therefore SG^2 - QG^2 = a^2(t^2+1)^2 + 4a^2 - a^2(t^2-1)^2 - 4a^2$

$\therefore SG^2 - QG^2 = a^2(t^4 + 1 + 2t^2) - a^2(t^4 + 1 - 2t^2)$

$\therefore SG^2 - QG^2 = a^2t^4 + a^2 + 2a^2t^2 - a^2t^4 - a^2 + 2a^2t^2$

$\therefore SG^2 - QG^2 = a^2t^4 + a^2 + 2a^2t^2 - a^2t^4 - a^2 + 2a^2t^2$

$$\therefore SG^2 - QG^2 = 4a^2t^2 \quad (71.4)$$

Equating eqns. (71.3) & (71.4)

$SG^2 - QG^2 = PQ^2$

$$PQ^2 + QG^2 = SG^2 \quad (71.5)$$

Eqn. (71.5) is mathematical expression of the theorem.

THEOREM- 72:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PM \times PS = OM \times TS}$

Derivations for proof of the theorem

$$\text{Eqn. (A.20)} \Longrightarrow PM = at\sqrt{t^2+1} \quad (72.1)$$

$$\text{Eqn. (A.14)} \Longrightarrow PS = 2a\sqrt{t^2+1} \quad (72.2)$$

$$\text{Eqn. (A.17)} \Longrightarrow OM = at \quad (72.3)$$

$$\text{Eqn. (A.15)} \Longrightarrow TS = 2a(t^2+1) \quad (72.4)$$

Multiplying eqns. (72.1) & (72.2),

$PM \times PS = \left(at\sqrt{t^2+1}\right) \times \left(2a\sqrt{t^2+1}\right)$

$$\therefore PM \times PS = 2a^2t(t^2+1) \quad (72.5)$$

Multiplying eqns. (72.3) & (72.4),

$$\mathrm{OM \times TS = at \times 2a(t^2 + 1)}$$

$$\therefore \mathrm{OM \times TS = 2a^2t(t^2 + 1)} \quad (72.6)$$

Equating eqns. (72.5) & (72.6)

$$\mathrm{PM \times PS = OM \times TS} \quad (72.7)$$

Eqn. (72.7) is mathematical expression of the theorem.

THEOREM- 73:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{PQ \times AN = AF \times FQ}$

Derivations for proof of the theorem

Referring fig. 1,

$$\mathrm{Eqn.\,(A.2) \Longrightarrow PQ = 2at} \quad (73.1)$$

$$\mathrm{Eqn.\,(B.1) \Longrightarrow AN = \frac{a(t^2-1)}{t}} \quad (73.2)$$

$$\mathrm{Eqn.\,(A.19) \Longrightarrow AF = 2a} \quad (73.3)$$

$$\mathrm{Eqn.\,(A.7) \Longrightarrow FQ = a(t^2 - 1)} \quad (73.4)$$

Multiplying eqns. (73.1) & (73.2),

$$\mathrm{PQ \times AN = 2at \times \left(\frac{a(t^2-1)}{t}\right)}$$

$$\therefore \mathrm{PQ \times AN = 2a^2(t^2 - 1)} \quad (73.5)$$

Multiplying eqns. (73.3) & (73.4),

$$\mathrm{AF \times FQ = 2a \times a(t^2 - 1)}$$

$\therefore \mathrm{AF} \times \mathrm{FQ}$
$= 2a^2(t^2 - 1)$ (73.6)

Equating eqns. (73.5) & (73.6)

$\mathrm{PQ} \times \mathrm{AN}$
$= \mathrm{AF} \times \mathrm{FQ}$ (73.7)

Eqn. (73.7) is mathematical expression of the theorem.

THEOREM- 74:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{AN \times OT = AT \times OM}$

Derivations for proof of the theorem

Referring fig. 2,

$\mathrm{Eqn.}\,(\mathrm{B.1}) \Longrightarrow \mathrm{AN}$
$= \frac{a(t^2 - 1)}{t}$ (74.1)

$\mathrm{Eqn.}\,(\mathrm{A.11}) \Longrightarrow \mathrm{OT}$
$= at^2$ (74.2)

$\mathrm{Eqn.}\,(\mathrm{A.21}) \Longrightarrow \mathrm{AT}$
$= a(t^2$
$- 1)$ (74.3)

$\mathrm{Eqn.}\,(\mathrm{A.17}) \Longrightarrow \mathrm{OM}$
$= at$ (74.4)

Multiplying eqns. (74.1) & (74.2),

$$\mathrm{AN} \times \mathrm{OT} = \frac{a(t^2 - 1)}{t} \times at^2$$

$\therefore \mathrm{AN} \times \mathrm{OT}$
$= a^2t(t^2 - 1)$ (74.5)

Multiplying eqns. (74.3) & (74.4),

$$\mathrm{AT} \times \mathrm{OM} = a(t^2 - 1) \times at$$

$\therefore \mathrm{AT} \times \mathrm{OM}$
$= a^2t(t^2 - 1)$ (74.6)

Equating eqns. (74.5) & (74.6)

$$\mathrm{AN} \times \mathrm{OT} = \mathrm{AT} \times \mathrm{OM} \quad (74.7)$$

Eqn. (74.7) is mathematical expression of the theorem.

THEOREM- 75:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PT \times PS = 4 \times PM \times FM}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn.}\,(\text{A}.13) \Longrightarrow \mathrm{PT} = 2\mathrm{at}\sqrt{\mathrm{t}^2+1} \quad (75.1)$$

$$\text{Eqn.}\,(\text{A}.14) \Longrightarrow \mathrm{PS} = 2\mathrm{a}\sqrt{\mathrm{t}^2+1} \quad (75.2)$$

$$\text{Eqn.}\,(\text{A}.20) \Longrightarrow \mathrm{PM} = \mathrm{at}\sqrt{\mathrm{t}^2+1} \quad (75.3)$$

$$\text{Eqn.}\,(\text{A}.18) \Longrightarrow \mathrm{FM} = \mathrm{a}\sqrt{\mathrm{t}^2+1} \quad (75.4)$$

Multiplying eqns. (75.1) & (75.2),

$$\mathrm{PT} \times \mathrm{PS} = \left(2\mathrm{at}\sqrt{\mathrm{t}^2+1}\right) \times \left(2\mathrm{a}\sqrt{\mathrm{t}^2+1}\right)$$

$$\therefore \mathrm{PT} \times \mathrm{PS} = 4\mathrm{a}^2\mathrm{t}(\mathrm{t}^2+1) \quad (75.5)$$

Multiplying eqns. 4, (75.3) & (75.4),

$$4 \times \mathrm{PM} \times \mathrm{FM} = 4\left(\mathrm{at}\sqrt{\mathrm{t}^2+1}\right) \times \left(\mathrm{a}\sqrt{\mathrm{t}^2+1}\right)$$

$$\therefore 4 \times \mathrm{PM} \times \mathrm{FM} = 4\mathrm{a}^2\mathrm{t}(\mathrm{t}^2+1) \quad (75.6)$$

Equating eqns. (75.5) & (75.6),

$$\mathrm{PT} \times \mathrm{PS} = 4 \times \mathrm{PM} \times \mathrm{FM} \quad (75.7)$$

Eqn. (75.7) is mathematical expression of the theorem.

THEOREM- 76:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{PT \times FN = PS \times FT}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn.}\,(A.13) \Longrightarrow PT = 2at\sqrt{t^2+1} \tag{76.1}$$

$$\text{Eqn.}\,(B.2) \Longrightarrow FN = \frac{a(t^2+1)}{t} \tag{76.2}$$

$$\text{Eqn.}\,(A.14) \Longrightarrow PS = 2a\sqrt{t^2+1} \tag{76.3}$$

$$\text{Eqn.}\,(A.16) \Longrightarrow FT = a(t^2+1) \tag{76.4}$$

Multiplying eqns. (76.1) & (76.2)

$$PT \times FN = 2at\sqrt{t^2+1} \times \left(\frac{a(t^2+1)}{t}\right)$$

$$\therefore PT \times FN = 2a^2(t^2+1)\sqrt{t^2+1} \tag{76.5}$$

Multiplying eqns. (76.3) & (76.4)

$$PS \times FT = 2a^2(t^2+1)\sqrt{t^2+1} \tag{76.6}$$

Equating eqns. (76.5) & (76.6),

$$PT \times FN = PS \times FT \tag{76.7}$$

Eqn. (76.7) is mathematical expression of the theorem.

THEOREM- 77:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{FX \times FY = GH \times FJ = a \times XY}$

Derivations for proof of the theorem

Referring fig. 5,

$$\text{Eqn. (D.18)} \Longrightarrow \mathrm{FX} = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}-1} \tag{77.1}$$

$$\text{Eqn. (D.19)} \Longrightarrow \mathrm{FY} = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}+1} \tag{77.2}$$

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{GH} = 2 \times \mathrm{FG} = 4a \tag{77.3}$$

$$\text{Eqn. (D.5)} \Longrightarrow \mathrm{FJ} = \frac{a(t^2+1)}{t^2} \tag{77.4}$$

$$\text{Eqn. (D.20)} \Longrightarrow \mathrm{XY} = \frac{4a(t^2+1)}{t^2} \tag{77.5}$$

Multiplying eqns. (77.1) & (77.2)

$$\mathrm{FX} \times \mathrm{FY} = \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}-1} \times \frac{2a\sqrt{t^2+1}}{\sqrt{t^2+1}+1}$$

$$\therefore \mathrm{FX} \times \mathrm{FY} = \frac{4a^2(t^2+1)}{t^2+1-1}$$

$$\therefore \mathrm{FX} \times \mathrm{FY} = \frac{4a^2(t^2+1)}{t^2} \tag{77.6}$$

Multiplying eqns. (77.3) & (77.4)

$$\mathrm{GH} \times \mathrm{FJ} = (4a) \times \left(\frac{a(t^2+1)}{t^2}\right)$$

$$\therefore \mathrm{GH} \times \mathrm{FJ} = \frac{4a^2(t^2+1)}{t^2} \tag{77.7}$$

$$a \times \text{Eqn.}(D.20) \Rightarrow \frac{4a^2(t^2+1)}{t^2} \quad (77.8)$$

Equating eqns. (77.7) & (77.8),

$$FX \times FY = GH \times FJ = a \times XY \quad (77.9)$$

Eqn. (77.9) is mathematical expression of the theorem.

THEOREM- 78:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{TS = 2 \times FP = 2 \times FT = 2 \times FS}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn.}(A.15) \Rightarrow TS = 2a(t^2+1) \quad (78.1)$$

$$\text{Eqn.}(A.9) \Rightarrow FP = a(t^2+1)$$

$$\therefore 2 \times FP = 2a(t^2+1) \quad (78.2)$$

$$\text{Eqn.}(A.16) \Rightarrow FT = a(t^2+1)$$

$$\therefore 2 \times FT = a(t^2+1) \quad (78.3)$$

$$\text{Eqn.}(A.23) \Rightarrow FS = a(t^2+1)$$

$$\therefore 2 \times FS = a(t^2+1) \quad (78.4)$$

Equating eqns. (78.1), (78.2), (78.3), (78.4),

$$TS = 2 \times FP = 2 \times FT = 2 \times FS \quad (78.5)$$

Eqn. (78.5) is mathematical expression of the theorem.

THEOREM- 79:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{PT = 2 \times QM = 2 \times FC}$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (A. 13)} \Longrightarrow \text{PT} = 2\text{at}\sqrt{\text{t}^2 + 1} \tag{79.1}$$

$$\text{Eqn. (A. 22)} \Longrightarrow \text{QM} = \text{at}\sqrt{\text{t}^2 + 1} \tag{79.2}$$

$$\text{Eqn. (C. 2)} \Longrightarrow \text{FC} = \text{at}\sqrt{\text{t}^2 + 1} \tag{79.3}$$

Equating eqns. (79.1), (79.2) & (79.3)

$$\text{PT} = 2 \times \text{QM} = 2 \times \text{FC} \tag{79.4}$$

Eqn. (79.4) is mathematical expression of the theorem.

THEOREM- 80:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{SQ = AF = 2 \times OA = FG = FH}$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 6)} \Longrightarrow \text{SQ} = 2\text{a} \tag{80.1}$$

$$\text{Eqn. (A. 19)} \Longrightarrow \text{AF} = 2\text{a} \tag{80.2}$$

We know that

$$\text{OA} = \text{a} \tag{80.3}$$

$$\text{Eqn. (A. 5)} \Longrightarrow \text{FG} = \text{GH} = \text{a} \tag{80.4}$$

Equating the above eqns. (1.1), (1.2), (1.3) & (1.4),

$$\mathrm{SQ} = \mathrm{AF} = 2 \times \mathrm{OA} = \mathrm{FG} = \mathrm{FH} \quad (10.5)$$

Eqn. (1.5) is mathematical expression of the theorem.

THEOREM- 81:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{FN - AN = TQ \times OW = \frac{RK}{2} = AF = SQ = 2a}$$

Derivations for proof of the theorem

$$\text{Eqn. (A.19)} \Longrightarrow \mathrm{FN} = \mathrm{a}(\mathrm{t}^2 + 1) \quad (81.1)$$

$$\text{Eqn. (A.5)} \Longrightarrow \mathrm{AN} = \mathrm{a}(\mathrm{t}^2 - 1) \quad (81.2)$$

Subtracting eqn. (81.2) from (81.1),

$$\mathrm{FN} - \mathrm{AN} = \mathrm{a}(\mathrm{t}^2 + 1) - \mathrm{a}(\mathrm{t}^2 - 1)$$

$$\therefore \mathrm{FN} - \mathrm{AN} = \mathrm{at}^2 + \mathrm{a} - \mathrm{at}^2 + \mathrm{a}$$

$$\therefore \mathrm{FN} - \mathrm{AN} = 2\mathrm{a} \quad (81.3)$$

$$\text{Eqn. (A.3)} \Longrightarrow \mathrm{AF} = 2\mathrm{a} \quad (81.4)$$

$$\text{Eqn. (A.6)} \Longrightarrow \mathrm{SQ} = 2\mathrm{a} \quad (81.5)$$

$$\text{Eqn. (C.5)} \Longrightarrow \mathrm{RK} = 4\mathrm{a}$$

$$\therefore \frac{\mathrm{RK}}{2} = 2\mathrm{a} \quad (81.6)$$

$$\text{Eqn. (A.12)} \Longrightarrow \mathrm{TQ} = 2\mathrm{at}^2 \quad (81.7)$$

$$\text{Eqn. (D.11)} \Longrightarrow \mathrm{OW} = \frac{1}{\mathrm{t}^2} \quad (81.8)$$

Multiplying eqns. (81.7) & (81.8)

$$TQ \times OW = 2at^2 \times \frac{1}{t^2}$$

$$\therefore TQ \times OW = 2a \tag{81.9}$$

Equating the above eqns. (81.3), (81.9), (81.6), (81.4) & (81.5),

$$FN - AN = TQ \times OW = \frac{RK}{2} = AF = SQ = 2a \tag{81.10}$$

Eqn. (81.10) is mathematical expression of the theorem.

THEOREM- 82:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$$\boldsymbol{PM \times PN = OQ \times PJ = FP^2 = FS^2 = FT^2}$$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 20)} \Longrightarrow PM = at\sqrt{t^2 + 1} \tag{82.1}$$

$$\text{Eqn. (B. 4)} \Longrightarrow PN = \frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t} \tag{82.2}$$

$$\text{Eqn. (A. 1)} \Longrightarrow OQ = at^2 \tag{82.3}$$

$$\text{Eqn. (D. 6)} \Longrightarrow PJ = \frac{a(t^2 + 1)^2}{t^2} \tag{82.4}$$

$$\text{Eqn. (A. 9)} \Longrightarrow FP = a(t^2 + 1)$$

$$\therefore FP^2 = a^2(t^2 + 1)^2 \tag{82.5}$$

$$\text{Eqn. (A. 23)} \Longrightarrow FS = a(t^2 + 1)$$

$$\therefore FS^2 = a^2(t^2 + 1)^2 \quad (82.6)$$

$$\text{Eqn. (A.16)} \Longrightarrow FT = a(t^2 + 1)$$

$$\therefore FT^2 = a^2(t^2 + 1)^2 \quad (82.7)$$

Multiplying eqns. (82.1) & (82.2),

$$PM \times PN = \left(at\sqrt{t^2 + 1}\right) \times \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore PM \times PN = a^2(t^2 + 1)^2 \quad (82.8)$$

Multiplying eqns. (82.3) & (82.4),

$$OQ \times PJ = at^2 \times \left(\frac{a(t^2 + 1)^2}{t^2}\right)$$

$$\therefore OQ \times PJ = a^2(t^2 + 1)^2 \quad (82.9)$$

Equating eqns. (82.5), (82.6), (82.7), (82.8) & (82.9)

$$PM \times PN = OQ \times PJ = FP^2 = FS^2 = FT^2 \quad (82.10)$$

Eqn. (82.10) is mathematical expression of the theorem.

THEOREM- 83:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$$\boldsymbol{N\mu \times J\mu = M\mu \times O\mu = NM^2}$$

Derivations for proof of the theorem

Referring fig. 6,

$$\text{Eqn. (D.26)} \Longrightarrow N\mu = a\sqrt{t^2 + 1} \quad (83.1)$$

$$\text{Eqn. (D.27)} \Longrightarrow J\mu = \frac{a\sqrt{t^2+1}}{t^2} \quad (83.2)$$

$$\text{Eqn. (D.24)} \Longrightarrow M\mu = \frac{a(t^2+1)}{t} \quad (83.3)$$

$$\text{Eqn. (D.27)} \Longrightarrow O\mu = \frac{a}{t} \quad (83.4)$$

$$\text{Eqn. (B.11)} \Longrightarrow NM = \frac{a\sqrt{t^2+1}}{t}$$

$$\therefore NM^2 = \frac{a^2(t^2+1)}{t^2} \quad (83.5)$$

Multiplying eqns. (83.1) & (83.2)

$$N\mu \times J\mu = a\sqrt{t^2+1} \times \left(\frac{a\sqrt{t^2+1}}{t^2}\right)$$

$$\therefore N\mu \times J\mu = \frac{a^2(t^2+1)}{t^2} \quad (83.6)$$

Multiplying eqns. (83.3) & (83.4)

$$M\mu \times O\mu = \left(\frac{a(t^2+1)}{t}\right) \times \frac{a}{t}$$

$$\therefore M\mu \times O\mu = \frac{a^2(t^2+1)}{t^2} \quad (83.7)$$

Equating eqns. (83.6), (83.7) & (83.5),

$$N\mu \times J\mu = M\mu \times O\mu = NM^2 \quad (83.8)$$

Eqn. (83.8) is mathematical expression of the theorem.

THEOREM- 84:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$\boldsymbol{QM \times J\mu = NM \times N\mu}$

Derivations for proof of the theorem

Referring fig. 6,

Eqn. (A. 17) $\Longrightarrow$ QM

$$= at\sqrt{t^2+1} \quad (84.1)$$

Eqn. (D. 27) $\Longrightarrow$ Jμ

$$= \frac{a\sqrt{t^2+1}}{t^2} \quad (84.2)$$

Eqn. (B. 11) $\Longrightarrow$ NM

$$= \frac{a\sqrt{t^2+1}}{t} \quad (84.3)$$

Eqn. (D. 26) $\Longrightarrow$ Nμ

$$= a\sqrt{t^2+1} \quad (84.4)$$

Multiplying eqns. (84.1) & (84.2)

$$QM \times J\mu = at\sqrt{t^2+1} \times \frac{a\sqrt{t^2+1}}{t^2}$$

$$\therefore OM \times J\mu = \frac{a^2(t^2+1)}{t} \quad (84.5)$$

Multiplying eqns. (84.3) & (84.4)

$$NM \times N\mu = \left(\frac{a\sqrt{t^2+1}}{t}\right) \times a\sqrt{t^2+1}$$

$$\therefore NM \times N\mu = \frac{a^2(t^2+1)}{t} \quad (84.6)$$

Equating eqns. (84.5) & (84.6),

$$QM \times J\mu = NM \times N\mu \quad (84.7)$$

Eqn. (84.7) is mathematical expression of the theorem.

THEOREM- 85:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\boldsymbol{VM \times M\mu \times OW = NM \times O\mu}$

Derivations for proof of the theorem

Referring fig. 2 & fig. 5,

$$\text{Eqn. (D. 23)} \Longrightarrow \text{VM} = \frac{at}{\sqrt{t^2+1}} \tag{85.1}$$

$$\text{Eqn. (D. 24)} \Longrightarrow \text{M}\mu = \frac{a(t^2+1)}{t} \tag{85.2}$$

$$\text{Eqn. (D. 25)} \Longrightarrow \text{O}\mu = \frac{a}{t} \tag{85.3}$$

$$\text{Eqn. (B. 11)} \Longrightarrow \text{NM} = \frac{a\sqrt{t^2+1}}{t} \tag{85.4}$$

$$\text{Eqn. (D. 11)} \Longrightarrow \text{OW} = \frac{1}{t^2} \tag{85.5}$$

Multiplying eqns. (85.1) & (85.2)

$$\text{VM} \times \text{M}\mu \times \text{OW} = \left(\frac{at}{\sqrt{t^2+1}}\right) \times \frac{a(t^2+1)}{t} \times \frac{1}{t^2}$$

$$\therefore \text{VM} \times \text{M}\mu = \frac{a^2\sqrt{t^2+1}}{t^2} \tag{85.6}$$

Multiplying eqns. (85.3) & (85.4)

$$\text{NM} \times \text{O}\mu = \left(\frac{a\sqrt{t^2+1}}{t}\right) \times \frac{a}{t}$$

$$\therefore \text{NM} \times \text{O}\mu = \frac{a^2\sqrt{t^2+1}}{t^2} \tag{85.7}$$

Equating eqns. (85.6) & (85.7),

$$\text{VM} \times \text{M}\mu \times \text{OW} = \text{NM} \times \text{O}\mu \tag{85.8}$$

Eqn. (85.8) is mathematical expression of the theorem.

THEOREM- 86:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$,

$$\boldsymbol{\frac{FP}{FM} = \frac{TS}{PS} = \frac{FT}{FM}}$$

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 9)} \Longrightarrow \mathrm{FP} = \mathrm{a(t^2 + 1)} \quad (86.1)$$

$$\text{Eqn. (A. 18)} \Longrightarrow \mathrm{FM} = \mathrm{a}\sqrt{\mathrm{t}^2 + 1} \quad (86.2)$$

$$\text{Eqn. (A. 15)} \Longrightarrow \mathrm{TS} = \mathrm{2a(t^2 + 1)} \quad (86.3)$$

$$\text{Eqn. (A. 14)} \Longrightarrow \mathrm{PS} = 2\mathrm{a}\sqrt{\mathrm{t}^2 + 1} \quad (86.4)$$

$$\text{Eqn. (A. 16)} \Longrightarrow \mathrm{FT} = \mathrm{a(t^2 + 1)} \quad (86.5)$$

Dividing eqn. (86.1) by (86.2),

$$\frac{\mathrm{FP}}{\mathrm{FM}} = \frac{\mathrm{a(t^2 + 1)}}{\mathrm{a}\sqrt{\mathrm{t}^2 + 1}}$$

$$\therefore \frac{\mathrm{FP}}{\mathrm{FM}} = \sqrt{\mathrm{t}^2 + 1} \quad (86.6)$$

Dividing eqn. (86.1) by (86.2)

$$\frac{\mathrm{TS}}{\mathrm{PS}} = \frac{\mathrm{2a(t^2 + 1)}}{2\mathrm{a}\sqrt{\mathrm{t}^2 + 1}}$$

$$\therefore \frac{\mathrm{TS}}{\mathrm{PS}} = \sqrt{\mathrm{t}^2 + 1} \quad (86.7)$$

Dividing eqn. (86.5) by (86.6)

$$\frac{FT}{FM} = \frac{a(t^2 + 1)}{a\sqrt{t^2 + 1}}$$

$$\therefore \frac{FT}{FM} = \sqrt{t^2 + 1} \qquad (86.8)$$

Equating eqns. (86.6), (86.7) & (86.8),

$$\frac{FP}{FM} = \frac{TS}{PS} = \frac{FT}{FM} \qquad (86.10)$$

Eqn. (86.10) is mathematical expression of the theorem.

THEOREM- 87:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{\frac{P\Omega}{S\Omega} = \frac{FS}{TA} = \frac{FJ}{FW} = \frac{FN}{AN} = \frac{FZ}{AN}}$$

Derivations for proof of the theorem

Referring fig. 3,

$$\text{Eqn. (E. 10)} \Longrightarrow P\Omega = \frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t^2} \qquad (87.1)$$

$$\text{Eqn. (E. 11)} \Longrightarrow S\Omega = \frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t^2} \qquad (87.2)$$

$$\text{Eqn. (A. 23)} \Longrightarrow FS = a(t^2 + 1) \qquad (87.3)$$

$$\text{Eqn. (A. 21)} \Longrightarrow TA = a(t^2 - 1) \qquad (87.4)$$

$$\text{Eqn. (D. 5)} \Longrightarrow FJ = \frac{a(t^2 + 1)}{t^2} \qquad (87.5)$$

$$\text{Eqn. (D. 10)} \Longrightarrow \text{FW} = \frac{a(t^2 - 1)}{t^2} \quad (87.6)$$

$$\text{Eqn. (B. 2)} \Longrightarrow \text{FN} = \frac{a(t^2 + 1)}{t^2} \quad (87.7)$$

$$\text{Eqn. (B. 1)} \Longrightarrow \text{AN} = \frac{a(t^2 - 1)}{t^2} \quad (87.8)$$

$$\text{Eqn. (B. 10)} \Longrightarrow \text{FZ} = \frac{a(t^2 + 1)}{t^2} \quad (87.9)$$

Dividing eqns. (87.1) by (87.2),

$$\frac{P\Omega}{S\Omega} = \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t^2}\right) \div \left(\frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t^2}\right)$$

$$\therefore \frac{P\Omega}{S\Omega} = \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t^2}\right) \times \left(\frac{t^2}{a(t^2 - 1)\sqrt{t^2 + 1}}\right)$$

$$\therefore \frac{P\Omega}{S\Omega} = \frac{t^2 + 1}{t^2 - 1} \quad (87.10)$$

Dividing eqns. (87.3) by (87.4),

$$\frac{FS}{TA} = \frac{a(t^2 + 1)}{a(t^2 - 1)}$$

$$\therefore \frac{FS}{TA} = \frac{t^2 + 1}{t^2 - 1} \quad (87.11)$$

Dividing eqns. (87.5) by (87.6),

$$\frac{FJ}{FW} = \left(\frac{a(t^2 + 1)}{t^2}\right) \div \left(\frac{a(t^2 - 1)}{t^2}\right)$$

$$\therefore \frac{FJ}{FW} = \left(\frac{a(t^2 + 1)}{t^2}\right) \times \left(\frac{t^2}{a(t^2 - 1)}\right)$$

$$\therefore \frac{FJ}{FW} = \frac{t^2+1}{t^2-1} \quad (87.12)$$

Dividing eqns. (87.7) by (87.8),

$$\frac{FN}{AN} = \left(\frac{a(t^2+1)}{t}\right) \div \left(\frac{a(t^2-1)}{t}\right)$$

$$\therefore \frac{FN}{AN} = \frac{a(t^2+1)}{t} \times \frac{t}{a(t^2-1)}$$

$$\therefore \frac{FN}{AN} = \frac{t^2+1}{t^2-1} \quad (87.13)$$

Dividing eqns. (87.9) by (87.8),

$$\therefore \frac{FZ}{AN} = \left(\frac{a(t^2+1)}{t}\right) \div \left(\frac{a(t^2-1)}{t}\right)$$

$$\therefore \frac{FZ}{AN} = \frac{a(t^2+1)}{t} \times \frac{t}{a(t^2-1)}$$

$$\therefore \frac{FZ}{AN} = \frac{t^2+1}{t^2-1} \quad (87.14)$$

Equating the above eqn. (87.10), (87.11), (87.12), (87.13) & (87.14)

$$\frac{P\Omega}{S\Omega} = \frac{FS}{TA} = \frac{FJ}{FW} = \frac{FN}{AN} = \frac{FZ}{AN} \quad (87.15)$$

Eqn. (87.15) is mathematical expression of the theorem.

THEOREM- 88:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$$\boldsymbol{\frac{PS}{PT} = \frac{AN}{FQ} = \frac{OM}{OT} = \frac{AF}{PQ} = \frac{FG}{PQ} = \frac{SQ}{PQ} = \frac{FJ}{FZ} = \frac{PN}{JN} = \frac{PU}{UQ} = \frac{WI}{TW} = \frac{PQ}{TQ} = \frac{RK}{PR} = \frac{FM}{TM} = \frac{OQ}{\partial\sigma}}$$

Derivations for proof of the theorem

Referring fig. 1, fig. 3 & fig. 7,

$$\text{Eqn. (D. 14)} \Longrightarrow PS = 2a\sqrt{t^2+1} \tag{88.1}$$

$$\text{Eqn. (A. 13)} \Longrightarrow PT = 2at\sqrt{t^2+1} \tag{88.2}$$

$$\text{Eqn. (A. 11)} \Longrightarrow OM = at \tag{88.3}$$

$$\text{Eqn. (A. 17)} \Longrightarrow OT = at^2 \tag{88.4}$$

$$\text{Eqn. (A. 7)} \Longrightarrow AN = \frac{a(t^2-1)}{t} \tag{88.5}$$

$$\text{Eqn. (B. 1)} \Longrightarrow FQ = a(t^2-1) \tag{88.6}$$

$$\text{Eqn. (A. 2)} \Longrightarrow AF = 2a \tag{88.7}$$

$$\text{Eqn. (A. 6)} \Longrightarrow PQ = 2at \tag{88.8}$$

$$\text{Eqn. (A. 5)} \Longrightarrow FG = 2a \tag{88.9}$$

$$\text{Eqn. (A. 6)} \Longrightarrow SQ = 2a \tag{88.10}$$

$$\text{Eqn. (D. 5)} \Longrightarrow FJ = \frac{a(t^2+1)}{t^2} \tag{88.11}$$

$$\text{Eqn. (B. 10)} \Longrightarrow FZ = \frac{a(t^2+1)}{t} \tag{88.12}$$

$$\text{Eqn. (B. 4)} \Longrightarrow PN = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \tag{88.13}$$

$$\text{Eqn. (B. 8)} \Longrightarrow JN = \frac{a(t^2+1)\sqrt{t^2+1}}{t^2} \tag{88.14}$$

$$\text{Eqn. (B. 6)} \Longrightarrow \text{PU} = \frac{2at}{\sqrt{t^2+1}} \tag{88.15}$$

$$\text{Eqn. (B. 5)} \Longrightarrow \text{UQ} = \frac{2at^2}{\sqrt{t^2+1}} \tag{88.16}$$

$$\text{Eqn. (D. 15)} \Longrightarrow \text{WI} = \frac{a(t^4+1)}{t^3} \tag{88.17}$$

$$\text{Eqn. (D. 13)} \Longrightarrow \text{TW} = \frac{a(t^4+1)}{t^2} \tag{88.18}$$

$$\text{Eqn. (A. 12)} \Longrightarrow \text{TQ} = 2at^2 \tag{88.19}$$

$$\text{Eqn. (C. 5)} \Longrightarrow \text{RK} = 4a \tag{88.20}$$

$$\text{Eqn. (A. 21)} \Longrightarrow \text{PR} = 4at \tag{88.21}$$

$$\text{Eqn. (A. 18)} \Longrightarrow \text{FM} = a\sqrt{t^2+1} \tag{88.22}$$

$$\text{Eqn. (A. 19)} \Longrightarrow \text{TM} = at\sqrt{t^2+1} \tag{88.23}$$

$$\text{Eqn. (A. 1)} \Longrightarrow \text{OQ} = at^2 \tag{88.24}$$

$$\text{Eqn. (E. 22)} \Longrightarrow \partial\sigma = at^3 \tag{88.25}$$

Dividing eqn. (88.1) by (88.2)

$$\frac{PS}{PT} = \frac{2a\sqrt{t^2+1}}{2at\sqrt{t^2+1}}$$

$$\frac{PS}{PT} = \frac{1}{t} \tag{88.26}$$

Dividing eqn. (88.3) by (88.4)

$$\frac{AN}{FQ}=\frac{\frac{a(t^2-1)}{t}}{a(t^2-1)}$$

$$\therefore \frac{AN}{FQ}=\frac{a(t^2-1)}{t}\times\frac{1}{a(t^2-1)}$$

$$\therefore \frac{AN}{FQ}=\frac{1}{t} \qquad (88.27)$$

Dividing eqn. (88.5) by (88.6)

$$\frac{OM}{OT}=\frac{at}{at^2}$$

$$\therefore \frac{OM}{OT}=\frac{1}{t} \qquad (88.28)$$

Dividing eqn. (88.7) by (88.8)

$$\frac{AF}{PQ}=\frac{2a}{2at}$$

$$\therefore \frac{AF}{PQ}=\frac{1}{t} \qquad (88.29)$$

Dividing eqn. (88.9) by (88.8)

$$\frac{FG}{PQ}=\frac{1}{t} \qquad (88.30)$$

Dividing eqn. (88.10) by (88.8)

$$\frac{SQ}{PQ}=\frac{2a}{2at}$$

$$\therefore \frac{SQ}{PQ}=\frac{1}{t} \qquad (88.31)$$

Dividing eqn. (88.11) by (88.12)

$$\frac{FJ}{FZ} = \left(\frac{a(t^2+1)}{t^2}\right) \div \left(\frac{a(t^2+1)}{t}\right)$$

$$\therefore \frac{FJ}{FZ} = \left(\frac{a(t^2+1)}{t^2}\right) \times \left(\frac{t}{a(t^2+1)}\right)$$

$$\therefore \frac{FJ}{FZ} = \frac{1}{t} \quad (88.32)$$

Dividing eqn. (88.13) by (88.14)

$$\frac{PN}{JN} = \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t}\right) \div \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t^2}\right)$$

$$\therefore \frac{PN}{JN} = \left(\frac{a(t^2+1)\sqrt{t^2+1}}{t}\right) \times \left(\frac{t^2}{a(t^2+1)\sqrt{t^2+1}}\right)$$

$$\therefore \frac{PN}{JN} = \frac{1}{t} \quad (88.33)$$

Dividing eqns. (88.15) by (88.16),

$$\frac{PU}{UQ} = \left(\frac{2at}{\sqrt{t^2+1}}\right) \div \left(\frac{2at^2}{\sqrt{t^2+1}}\right)$$

$$\therefore \frac{PU}{UQ} = \left(\frac{2at}{\sqrt{t^2+1}}\right) \times \left(\frac{\sqrt{t^2+1}}{2at^2}\right)$$

$$\therefore \frac{PU}{UQ} = \frac{1}{t} \quad (88.34)$$

Dividing eqns. (88.17) by (88.18),

$$\frac{WI}{TW} = \left(\frac{a(t^4+1)}{t^3}\right) \div \left(\frac{a(t^4+1)}{t^2}\right)$$

$$\therefore \frac{WI}{TW} = \left(\frac{a(t^4+1)}{t^3}\right) \times \left(\frac{t^2}{a(t^4+1)}\right)$$

$$\therefore \frac{WI}{TW} = \frac{1}{t} \qquad (88.35)$$

Dividing eqn. (88.8) by (88.19)

$$\frac{PQ}{TQ} = \frac{2at}{2at^2}$$

$$\therefore \frac{PQ}{TQ} = \frac{1}{t} \qquad (88.36)$$

Dividing eqn. (88.20) by (88.21)

$$\frac{RK}{PR} = \frac{4a}{4at}$$

$$\therefore \frac{RK}{PR} = \frac{1}{t} \qquad (88.37)$$

Dividing eqn. (88.22) by (88.23)

$$\frac{FM}{TM} = \frac{a\sqrt{t^2+1}}{at\sqrt{t^2+1}}$$

$$\therefore \frac{FM}{TM} = \frac{1}{t} \qquad (88.38)$$

Dividing eqn. (88.24) by (88.25)

$$\frac{OQ}{\partial\sigma} = \frac{at^2}{at^3}$$

$$\therefore \frac{OQ}{\partial\sigma} = \frac{1}{t} \qquad (88.39)$$

Equating eqns. (88.26), (88.27), (88.28), (88.29), (88.30), (88.31), (88.32), (88.33), (88.34), (88.35), (88.36), (88.37) (88.38) & (88.39),

$$\frac{PS}{PT}=\frac{AN}{FQ}=\frac{OM}{OT}=\frac{AF}{PQ}=\frac{FG}{PQ}=\frac{SQ}{PQ}=\frac{FJ}{FZ}=\frac{PN}{JN}=\frac{PU}{UQ}=\frac{WI}{TW}=\frac{PQ}{TQ}=\frac{RK}{PR}=\frac{FM}{TM}=\frac{OQ}{\partial\sigma} \quad (88.40)$$

Eqn. (88.40) is mathematical expression of the theorem.

THEOREM- 89:

In a parabola $\boldsymbol{y^2=4ax}$, the parametric equation of parabola is $(\boldsymbol{x},\boldsymbol{y})=(\boldsymbol{at^2},\boldsymbol{2at})$,

$$\boldsymbol{\frac{FQ}{FP}=\frac{TN}{PN}=\frac{TA}{FT}=\frac{AL}{FM}=\frac{PK\times FM}{RK\times FT}}$$

Derivations for proof of the theorem

Referring fig. 1, fig. 2 & fig. 7,

$$\text{Eqn. (A.7)} \Rightarrow FQ = a(t^2-1) \quad (89.1)$$

$$\text{Eqn. (A.9)} \Rightarrow FP = a(t^2+1) \quad (89.2)$$

$$\text{Eqn. (B.3)} \Rightarrow TN = \frac{a(t^2-1)\sqrt{t^2+1}}{t} \quad (89.3)$$

$$\text{Eqn. (B.4)} \Rightarrow PN = \frac{a(t^2+1)\sqrt{t^2+1}}{t} \quad (89.4)$$

$$\text{Eqn. (A.21)} \Rightarrow TA = a(t^2-1) \quad (89.5)$$

$$\text{Eqn. (A.16)} \Rightarrow FT = a(t^2+1) \quad (89.6)$$

$$\text{Eqn. (B.9)} \Rightarrow AL = \frac{a(t^2-1)}{\sqrt{t^2+1}} \quad (89.7)$$

$$\text{Eqn. (A.18)} \Rightarrow FM = a\sqrt{t^2+1} \quad (89.8)$$

$\text{Eqn. (B. 9)} \Longrightarrow \text{PK}$
$$= 4a\sqrt{t^2 - 1} \tag{89.9}$$

$\text{Eqn. (A. 18)} \Longrightarrow \text{RK}$
$$= 4a \tag{89.10}$$

Dividing eqn. (89.1) by (89.2)

$$\frac{FQ}{FP} = \frac{a(t^2 - 1)}{a(t^2 + 1)}$$

$$\therefore \frac{FQ}{FP} = \frac{t^2 - 1}{t^2 + 1} \tag{89.11}$$

Dividing eqn. (89.3) by (89.4)

$$\frac{TN}{PN} = \left(\frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t}\right) \div \left(\frac{a(t^2 + 1)\sqrt{t^2 + 1}}{t}\right)$$

$$\therefore \frac{TN}{PN} = \left(\frac{a(t^2 - 1)\sqrt{t^2 + 1}}{t}\right) \times \left(\frac{t}{a(t^2 + 1)\sqrt{t^2 + 1}}\right)$$

$$\therefore \frac{TN}{PN} = \frac{t^2 - 1}{t^2 + 1} \tag{89.12}$$

Dividing eqn. (89.5) by (89.6)

$$\frac{TA}{FT} = \frac{a(t^2 - 1)}{a(t^2 + 1)}$$

$$\therefore \frac{TA}{FT} = \frac{t^2 - 1}{t^2 + 1} \tag{89.13}$$

Dividing eqn. (89.7) by (89.8)

$$\frac{AL}{FM} = \left(\frac{a(t^2 - 1)}{\sqrt{t^2 + 1}}\right) \div a\sqrt{t^2 + 1}$$

$$\therefore \frac{AL}{FM} = \frac{a(t^2 - 1)}{\sqrt{t^2 + 1}} \times \frac{1}{a\sqrt{t^2 + 1}}$$

$$\therefore \frac{AL}{FM} = \frac{t^2 - 1}{t^2 + 1} \quad (89.14)$$

Multiplying eqns. (89.9) & (89.8)

$$PK \times FM = 4a\sqrt{t^2 - 1} \times a\sqrt{t^2 + 1}$$

$$\therefore PK \times FM = 4a^2\sqrt{t^2 - 1} \times \sqrt{t^2 + 1} \quad (89.15)$$

Multiplying eqns. (89.10) & (89.6)

$$RK \times FT = 4a \times a(t^2 + 1)$$

$$\therefore RK \times FT = 4a^2(t^2 + 1) \quad (89.16)$$

Dividing eqn. (89.15) by (89.16)

$$\frac{PK \times FM}{RK \times FT} = \frac{4a^2\sqrt{t^2 - 1} \times \sqrt{t^2 + 1}}{4a^2(t^2 + 1)}$$

$$\therefore \frac{PK \times FM}{RK \times FT} = \frac{\sqrt{t^2 - 1}}{\sqrt{t^2 + 1}} \quad (89.17)$$

Equating eqns. (89.11), (89.12), (89.13), (89.14) & (89.17)

$$\frac{FQ}{FP} = \frac{TN}{PN} = \frac{TA}{FT} = \frac{AL}{FM} = \frac{PK \times FM}{RK \times FT} \quad (89.18)$$

Eqn. (89.18) is mathematical expression of the theorem.

THEOREM- 90:

Referring fig. 1, In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$,

$\Delta\, \boldsymbol{TQM}$ & $\Delta\, \boldsymbol{TSZ}$ are similar.

Derivations for proof of the theorem

Referring fig. 1,

$$\text{Eqn. (A. 11)} \Longrightarrow OT = at^2 \quad (90.1)$$

$$\text{Eqn. (A. 1)} \Longrightarrow OQ = at^2 \quad (90.2)$$

$$\text{Eqn. (A. 16)} \Longrightarrow FT = a(t^2 + 1) \quad (90.3)$$

$$\text{Eqn. (A. 23)} \Longrightarrow FS = a(t^2 + 1) \quad (90.4)$$

Equating eqns. (90.1) & (90.2),

$$OT = OQ \quad (90.5)$$

Therefore, it is understand that Δ TQM is Isosceles triangle. A property of Isosceles triangle is that

$$\text{If} \angle QFP = \theta°, \angle QTM = \angle TQM = \frac{\theta°}{2}$$

Equating eqns. (90.3) & (90.4),

$$FT = FS \quad (90.6)$$

Therefore, it is understand that Δ TSZ is Isosceles triangle. A property of Isosceles triangle is that

$$\text{If} \angle QFP = \theta°, \angle TSZ = \angle STZ = \frac{\theta°}{2}$$

Comparing Δ TQM & Δ TSZ, point T is a common vertex and $\angle QTM = \angle STZ$ and $\angle TQM = \angle TSZ$.

Therefore, Δ TQM & Δ TSZ are similar.

THEOREM- 91:

Referring fig. 4, Let a parabola be given with focus F. Suppose a tangent drawn at a point P and PT is segment of the tangent. If from the focus F, a perpendicular FM is drawn to the tangent line PT, then:

(i) FM is the perpendicular bisector of the tangent segment PT.

(ii) FM is also the internal angle bisector of the ∠PFT.

Derivations for proof of the theorem

In a parabola $y^2 = 4ax$, the parametric equation of parabola is $(x, y) = (at^2, 2at)$, Referring fig. 1, Let a parabola be given with focus F.

Suppose PT is a tangent to the parabola at a point P. If from the focus F, a perpendicular FM is drawn to the tangent line PT, then:

(i) FM is the perpendicular bisector of the tangent segment PT.

(ii) FM is also the internal angle bisector [7] of the ∠ PFT.

THEOREM- 92:

Referring fig. 4, Let a parabola be given with focus F. Suppose PS is a normal drawn to the parabola at a point P. If from the focus F, a perpendicular line FC is drawn to the normal line PS, then:

(i) FC is the perpendicular bisector of the segment of normal (PS).

(ii) FC is also the internal angle bisector of the ∠ PFS.

Derivations for proof of the theorem

In a parabola $y^2 = 4ax$, the parametric equation of parabola is $(x, y) = (at^2, 2at)$,

Referring fig. 3, Let a parabola be given with focus F.

Suppose PS is a normal to the parabola at a point P. If from the focus F, a perpendicular FC is drawn to the normal line PS, then:

(i) FC is the perpendicular bisector of the normal segment PS.

(ii) FC is also the internal angle bisector of the ∠ PFS.

THEOREM- 93:

Let PT be the tangent drawn at a point P on the parabola $\boldsymbol{y^2 = 4ax}$. Let Q be the projection of P on the axis of symmetry of the parabola. Then, the circle having PT as its diameter touches the axis of symmetry at the point Q.

Derivations for proof of the theorem

Referring circle-A in fig. 8, let PT be the tangent drawn at a point P on the parabola $y^2 = 4ax$. Let Q be the projection of P on the axis of symmetry of the parabola. Then, the circle having PT as its diameter touches the axis of symmetry at the point Q.

Let P be the point on the parabola $y^2 = 4ax$ with parametric coordinates $P(at^2, 2at)$. Let PT be the tangent at P, and let Q be the projection of P on the axis of symmetry (the x-axis), so $Q = (at^2, 0)$. Then the circle having PT as diameter passes through Q (and also through T). Moreover, the circle meets the axis at the two points Q and T

1. Parametric data and tangent.
For $y^2 = 4ax$ take the parametric point $P(at^2, 2at)$.

The tangent at P has equation (standard parametric tangent)

$ty = x + at^2$

Its intersection with the axis $y = 0$ gives the point T. Putting $y = 0$ in the tangent:

$0 = x + at^2 \Longrightarrow x = -at^2,$

So, $T = (-at^2, 0)$.

2. Equation of the circle with diameter PT.
A circle with diameter end points $P(x_1, y_1)$ and $T(x_2, y_2)$ satisfies

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

Substituting $(x_1, y_1) = (at^2, 2at)$ and $(x_2, y_2) = (-at^2, 0)$ in above,

Then the circle equation is

$(x - at^2)(x + at^2) + (y - 2at)(y - 0) = 0,$

$\therefore x^2 - a^2t^4 + y(y - 2at) = 0.$

3. Verify that $Q = (at^2, 0)$ lies on the circle.
Substitute $x = at^2$, $y = 0$:

$(at^2)^2 - a^2t^4 + 0(0 - 2at) = a^2t^4 - a^2t^4 = 0.$

So, Q satisfies the circle equation-the circle passes through Q.

Likewise, substituting $T = (-at^2, 0)$ also gives zero, so the circle passes through T as well.

4. Intersection type with the axis $y = 0$.
Restrict the circle equation to $y = 0$:

$x^2 - a^2t^4 = 0 \Longrightarrow x = \pm at^2.$

These two solutions correspond exactly to $x = at^2$ (point Q) and $x = -at^2$ (point T). Because there are two distinct intersection points (unless $t = 0$), the line $y = 0$ is a secant of the circle, not a tangent. Tangency would require a double root (a repeated intersection), i.e. $at^2 = -at^2$, which forces $t = 0$. So only in the degenerate case $t = 0$ (when P is the vertex and $P = Q = T = (0,0)$ does the circle become tangent to the axis.

Therefore, the correct statement is that the circle with diameter PT passes through the projection Q (and through T), but it is not generally tangent to the axis at Q.

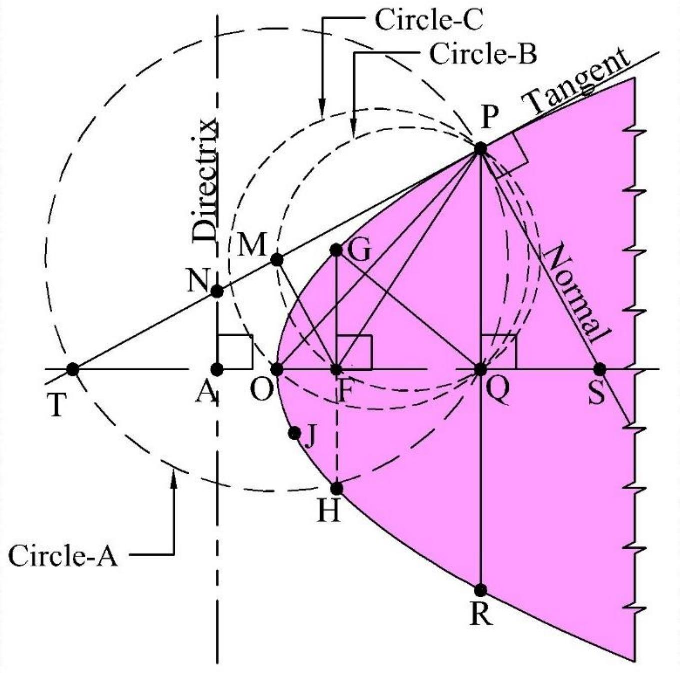

Fig. 8

THEOREM- 94:

Let P be a point on the parabola $\boldsymbol{y^2 = 4ax}$, and let the tangent at P be drawn. Let $\boldsymbol{F(a, 0)}$ be the focus of the parabola, and let the line through F parallel to the normal at P intersect the tangent at M. Let Q be the projection of P on the axis of symmetry (the x-axis). Then, the Quadrilateral which is formed by joining points F, Q, P & M is cyclic.

Derivations for proof of the theorem

Referring circle-B in fig. 8,

Take the parametric coordinates of P: P $(at^2, 2at)$.

A quadrilateral is cyclic [8] (i.e., its four vertices lie on the circumference of a circle) if it satisfies the condition that the sum of each pair of opposite angles is 180°.

If $\angle PFQ = \theta°$,

$$\angle FPM = \angle MPF = \angle PTQ = \frac{\theta°}{2} \quad (94.1)$$

Referring fig. 6, we know that

$$\angle PMF = 90° \quad (94.2)$$

$$\angle PQF = 90° \quad (94.3)$$

$$\angle MPQ = \angle MPF + \angle FPQ$$

$$\therefore \angle MPQ = \frac{\theta°}{2} + (90° - \theta°)$$

$$\therefore \angle MPQ = 90° - \frac{\theta°}{2} \quad (94.4)$$

Referring fig. 8, we know that

$$\angle MFQ = \angle MFP + \angle PFQ$$

$$\therefore \angle MFQ = \left(90 - \frac{\theta°}{2}\right) + \theta°$$

$$\therefore \angle MFQ = 90 + \frac{\theta°}{2} \quad (94.5)$$

Adding eqns. (94.2) & (94.3),

$$\angle PMF + \angle PQF = 180° \quad (94.6)$$

Adding eqns. (94.4) & (94.5),

$$\angle MPQ + \angle MFQ = \left(90° - \frac{\theta°}{2}\right) + \left(90° + \frac{\theta°}{2}\right)$$

$$\angle MPQ + \angle MFQ = 180° \quad (94.7)$$

Equating eqns. (94.6) & (94.7),

$$\angle PMF + \angle PQF = \angle MPQ + \angle MFQ = 180° \quad (94.8)$$

Hence, the condition for cyclic quadrilateral is satisfied. Therefore, the quadrilateral FQPM is cyclic.

THEOREM- 95:

Let P be a point on the parabola $\boldsymbol{y^2 = 4ax}$, with vertex O (0,0)) and focus F(a,0). Let OP be the line segment joining O and P, and let Q be the projection of P on the axis of symmetry (the x-axis). Then the circle having OP as its diameter touches the axis of symmetry at the points O and Q.

Derivations for proof of the theorem

Referring circle-C in fig. 8,

1. Parametric coordinates of P:
On the parabola $y^2 = 4ax$ let P $(at^2, 2at)$, $t \in \mathbb{R}$.

The vertex is O (0,0).
Projection of P on the axis (x-axis) is Q $(at^2, 0)$.

2. Equation of the circle with diameter OP:
The equation of a circle with diameter endpoints $(x_1, y_1), (x_2, y_2)$ is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$

Here, $O(x_1, y_1) = (0,0), P(x_2, y_2) = (at^2, 2at).$
Thus, the circle is: $(x - 0)(x - at^2) + (y - 0)(y - 2at) = 0,$

i.e. $x(x - at^2) + y(y - 2at) = 0.$.

Expanding:

$x^2 - at^2x + y^2 - 2aty = 0.$

3. Intersection with the axis of symmetry $(y = 0)$:
On the axis, put $y = 0$:

$x^2 - at^2x = 0,$

$x(x - at^2) = 0.$

So, $x = 0$ or $x = at^2$.

Thus, the points of intersection are $O(0,0)$ and $Q(at^2, 0)$.

Tangency at O and Q:
For tangency, the axis must touch the circle at those points.

At O(0,0): substitute into circle equation → holds. The circle meets axis only once there (double root), so tangent.

At Q(at^2,0): substitute into circle equation → holds. Similarly, it is a repeated root, so tangent at Q.

(Formally: When substituting $y = 0$, the quadratic in x gave roots 0 and at^2. These correspond to the two ends of the diameter. Because the axis passes through both endpoints of the diameter, it touches the circle at those endpoints, i.e., it is tangent there.)

Thus, the circle with diameter OP touches the axis of symmetry at the points O and Q.

THEOREM- 96

Let a tangent be drawn at a point P on the parabola $\boldsymbol{y^2 = 4ax}$, and let this tangent intersect the axis of symmetry at T. If a line is drawn through the focus F(a,0) perpendicular to the tangent, then that line bisects the segment of the tangent (PT).

Derivations for proof of the theorem

Referring fig.1,

1. Equation of the tangent at P

Parametric eqn. of parabola $y^2 = 4ax \implies P(x, y) = (at^2, 2at)$

Parametric eqn. of tangent at point P $\implies ty$

$$= x + at^2 \tag{96.1}$$

Slope of tangent at point P $\implies \frac{1}{t}$

2. Point T (intersection of the tangent with axis of symmetry)

Let Point T is intersection of the tangent and axis of symmetry

Eqn. of axis of symmetry $\implies y = 0$

Substituting $y = 0$ in eqn. (80.1),

$t \times (0) = x + at^2$

$\therefore x + at^2 = 0$

$\therefore x = -at^2$

$\therefore$ Co-ordinates of the point T $(x, y) \implies (-at^2, 0)$

3. Midpoint of line segment PT

Let midpoint of line segment PT is M.

The coordinates of midpoint M $(x, y) \implies \left(\frac{at^2 + (-at^2)}{2}, \frac{2at + 0}{2}\right)$

$\therefore$ M $(x, y) \implies (0, at)$

4. Line through focus perpendicular to the tangent

If slope of tangent at point P $= \frac{1}{t}$,

the slope of any line perpendicular to the tangent is $-\frac{1}{\left(\frac{1}{t}\right)} \implies -t$

Eqn. of line through the focus F(a, 0) with slope of $-t$

$y - 0 = -t(x - a)$

$\therefore y = -tx + at$

Evaluate this at $x = 0$ & $y = at$. Thus the line passes through the point $(0, at)$ which is exactly the midpoint M.

Therefore, the line through the focus perpendicular to the tangent meets PT at its midpoint, it bisects PT.

THEOREM- 97:

For the parabola $\boldsymbol{y^2 = 4ax}$, let a tangent be drawn at a point P. If a line parallel to the normal at P is drawn through the focus F(a,0), then this line intersects the tangent at the point where the tangent meets the vertical line drawn through the vertex of the parabola.

Derivations for proof of the theorem:

Referring fig. 1,

1. Equation of the tangent at P

Parametric eqn. of parabola $y^2 = 4ax \implies P(x, y) = (at^2, 2at)$

Parametric eqn. of tangent at point P $\implies ty = x + at^2$ (97.1)

2. Point M (intersection of the tangent with axis of symmetry)

Let Point M is intersection of the tangent and vertical line at O

Eqn. of vertical line at O(0,0) $\implies x = 0$

Substituting $x = 0$ in eqn. (81.1),

$t \times y = 0 + at^2$

$$\therefore y = \frac{at^2}{t} = at$$

Hence the tangent meets the vertical line at Q $(x, y) = (0, at)$

3. Equation of line parallel to normal the focus F (a,0)

The slope of tangent at P is $\frac{1}{t}$, so the slope of the normal at P is $-t$

The line through the focus F (a,0) with slope $-t$ is

$y = -t(x - a)$

Evaluate this by substituting $x = 0$

$y = -t(0 - a)$

$\therefore y = at$

Hence the line passes through the point M $(0, at)$.

Therefore, the line through the focus parallel to the normal at P meets the tangent at the same point where the tangent meets the vertical line $x = 0$.

THEOREM- 98:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{at^2}, \boldsymbol{2at})$, If two focal chords cut orthogonally,

$$\frac{\mathbf{1}}{\boldsymbol{PQ}} + \frac{\mathbf{1}}{\boldsymbol{RS}} = \frac{\mathbf{1}}{\boldsymbol{GH}}$$

Derivations for proof of the theorem

Referring fig. 9, F is focus, PQ and RS are pair focal chord of intersects orthogonally each other. GH is Latus rectum is equal to 4a. Let $\angle$ TFP = θ°, $\angle$ PFR = 90°. Let points K, L, M & N are projection of points P, R, Q, S on axis of symmetry respectively.

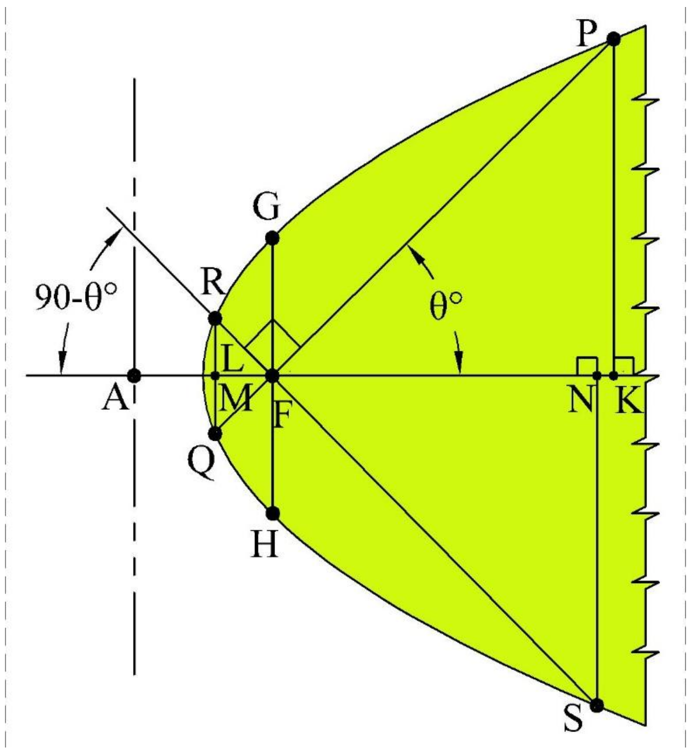

Fig. 9

$$\text{Eqn. (D.1)} \Longrightarrow \text{FP} = \frac{2a}{1-\cos(\theta°)} \tag{98.1}$$

If $\angle \text{TFP} = \theta°, \angle \text{TFQ} = 180° + \theta°$

Substituting the above value in eqn. (98.1),

$$\text{FQ} = \frac{2a}{1-\cos(180° + \theta°)}$$

$$\therefore \text{FQ} = \frac{2a}{1-[-\cos(\theta°)]}$$

$$\therefore \text{FQ} = \frac{2a}{1+\cos(\theta°)} \tag{98.2}$$

Referring fig. 9, $\text{PQ} = \text{FP} + \text{FQ}$

Substituting eqns. (98.1) & (98.2),

$$\text{PQ} = \left(\frac{2a}{1-\cos(\theta°)}\right) + \left(\frac{2a}{1+\cos(\theta°)}\right)$$

$$\therefore \text{PQ} = \frac{2a\big(1+\cos(\theta°)\big) + 2a\big(1-\cos(\theta°)\big)}{1-\cos^2(\theta°)}$$

$$\therefore \text{PQ} = \frac{2a\big(1+\cos(\theta°) + 1 - \cos(\theta°)\big)}{\sin^2(\theta°)}$$

$$\therefore \text{PQ} = \frac{2a\big(1+\cos(\theta°) + 1 - \cos(\theta°)\big)}{\sin^2(\theta°)}$$

$$\therefore \text{PQ} = \frac{2a(2)}{\sin^2(\theta°)}$$

$$\therefore \text{PQ} = \frac{4a}{\sin^2(\theta°)} \tag{98.3}$$

If $\angle \text{TFP} = \theta°, \angle \text{TFR} = 90° + \theta°$

Substituting the above value in eqn. (98.1),

$$\text{FR} = \frac{2a}{1-\cos(90° + \theta°)}$$

$$\therefore \mathrm{FR} = \frac{2a}{1 - [-\sin(\theta°)]}$$

$$\therefore \mathrm{FR} = \frac{2a}{1 + \sin(\theta°)} \tag{98.4}$$

If $\angle \mathrm{TFP} = \theta°, \angle \mathrm{TFS} = 270° + \theta°$

Substituting the above value in eqn. (98.1),

$$\mathrm{FS} = \frac{2a}{1 - \cos(270° + \theta°)}$$

$$\therefore \mathrm{FS} = \frac{2a}{1 - [\sin(\theta°)]}$$

$$\therefore \mathrm{FS} = \frac{2a}{1 - \sin(\theta°)} \tag{98.5}$$

Referring fig. 9, $\mathrm{RS} = \mathrm{FR} + \mathrm{FS}$

Substituting eqns. (98.4) & (98.5),

$$\mathrm{RS} = \left(\frac{2a}{1 + \sin(\theta°)}\right) + \left(\frac{2a}{1 - \sin(\theta°)}\right)$$

$$\therefore \mathrm{RS} = \frac{2a\big(1 - \sin(\theta°)\big) + 2a\big(1 + \sin(\theta°)\big)}{1 - \sin^2(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{2a\big(1 - \sin(\theta°) + 1 + \sin(\theta°)\big)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{2a(2)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{RS} = \frac{4a}{\cos^2(\theta°)} \tag{98.6}$$

Adding reciprocal of eqns. (98.3) & (98.6),

$$\frac{1}{\mathrm{PQ}} + \frac{1}{\mathrm{RS}} = \left(\frac{\sin^2(\theta°)}{4a}\right) + \left(\frac{\cos^2(\theta°)}{4a}\right)$$

$$\therefore \frac{1}{\mathrm{PQ}} + \frac{1}{\mathrm{RS}} = \frac{\sin^2(\theta°) + \cos^2(\theta°)}{4a}$$

$$\therefore \frac{1}{PQ} + \frac{1}{RS} = \frac{1}{4a}$$

We know that Latus rectum GH = 4a and substituting this value in above eqn..

$$\frac{1}{PQ} + \frac{1}{RS} = \frac{1}{GH} \qquad (98.7)$$

Eqn. (98.7) is mathematical expression of the theorem.

THEOREM- 99:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$, If two focal chords cut orthogonally each other, then

$$\boldsymbol{FK \times FM \times FL \times FN = (FG)^4}$$

Derivations for proof of the theorem

Referring fig. 9, F is focus, PQ and RS are pair focal chord of intersects orthogonally each other. GH is Latus rectum is equal to 4a. Let ∠ TFP = θ°, ∠ PFR = 90°. Let points K, L, M & N are projection of points P, R, Q, S on axis of symmetry respectively. PK ⊥ FK, SN ⊥ FM, RL ⊥ FK & QM ⊥ FM.

$$\text{Eqn.}\,(98.1) \Longrightarrow FP = \frac{2a}{1 - \cos(\theta°)} \qquad (99.1)$$

$$\text{Eqn.}\,(98.2) \Longrightarrow FQ = \frac{2a}{1 + \cos(\theta°)} \qquad (99.2)$$

$$\text{Eqn.}\,(98.3) \Longrightarrow FR = \frac{2a}{1 + \sin(\theta°)} \qquad (99.3)$$

$$\text{Eqn.}\,(98.4) \Longrightarrow FS = \frac{2a}{1 - \sin(\theta°)} \qquad (99.4)$$

Referring fig. 9,

$$FK = FP \times \cos(\theta°)$$

Substituting eqn. (99.1) in above,

$$\therefore FK = \frac{2a\cos(\theta°)}{1-\cos(\theta°)} \tag{99.5}$$

Referring fig. 9,

$$FM = FQ \times \cos(\theta°)$$

Substituting eqn. (99.2) in above,

$$\therefore FM = \frac{2a\cos(\theta°)}{1+\cos(\theta°)} \tag{99.6}$$

Referring fig. 9,

If $\angle KFP = \theta°, \angle LFR = 90° - \theta°$

$$FL = FR \times \cos(90 - \theta°)$$

Substituting eqn. (99.3) in above,

$$\therefore FL = \frac{2a \times \cos(90-\theta°)}{1+\sin(\theta°)}$$

$$\therefore FL = \frac{2a\sin(\theta°)}{1+\sin(\theta°)} \tag{99.7}$$

Referring fig. 9,

If $\angle KFP = \theta°, \angle NSF = 90° - \theta°$

$$FN = FS \times \cos(90 - \theta°)$$

Substituting eqn. (99.4) in above,

$$\therefore FN = \frac{2a \times \cos(90-\theta°)}{1-\sin(\theta°)}$$

$$\therefore FN = \frac{2a\sin(\theta°)}{1-\sin(\theta°)} \tag{99.8}$$

Multiplying eqns. (99.5), (99.6), (99.7) & (99.8),

$$FK \times FM \times FL \times FN = \left[\left(\frac{2a\cos(\theta°)}{1-\cos(\theta°)}\right) \times \left(\frac{2a\cos(\theta°)}{1+\cos(\theta°)}\right)\right] \times \left[\left(\frac{2a\sin(\theta°)}{1+\sin(\theta°)}\right) \times \left(\frac{2a\sin(\theta°)}{1-\sin(\theta°)}\right)\right]$$

$$\therefore \mathrm{FK} \times \mathrm{FM} \times \mathrm{FL} \times \mathrm{FN} = \left[\frac{4a^2\cos^2(\theta°)}{1-\cos^2(\theta°)}\right] \times \left[\frac{4a^2\sin^2(\theta°)}{1-\sin^2(\theta°)}\right]$$

$$\therefore \mathrm{FK} \times \mathrm{FM} \times \mathrm{FL} \times \mathrm{FN} = \frac{4a^2\cos^2(\theta°)}{\sin^2(\theta°)} \times \frac{4a^2\sin^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \mathrm{FK} \times \mathrm{FM} \times \mathrm{FL} \times \mathrm{FN} = 16 \times a^4$$

$$\therefore \mathrm{FK} \times \mathrm{FM} \times \mathrm{FL} \times \mathrm{FN} = (2a)^4$$

$$\therefore \mathrm{FK} \times \mathrm{FM} \times \mathrm{FL} \times \mathrm{FN} = (\mathrm{FG})^4 \qquad (99.9)$$

where, FG is semi latus-rectum.

Eqn. (99.9) is mathematical expression of the theorem.

THEOREM- 100:

In a parabola $\boldsymbol{y^2 = 4ax}$, the parametric equation of parabola is $(\boldsymbol{x, y}) = (\boldsymbol{at^2, 2at})$, If two focal chords cut orthogonally, then

$$\boldsymbol{PK \times QM = RL \times SN = (FG)^2}$$

Derivations for proof of the theorem

Referring fig. 9, F is focus, PQ and RS are pair focal chord of intersects orthogonally each other. GH is Latus rectum is equal to 4a. Let, $\angle \mathrm{TFP} = \theta°$, $\angle \mathrm{PFR} = 90°$. Let points K, L, M & N are projection of points P, R, Q, S on axis of symmetry respectively. PK ⊥ FK, SN ⊥ FM, RL ⊥ FK & QM ⊥ FM.

$$\text{Eqn. } (98.1) \Longrightarrow \mathrm{FP} = \frac{2a}{1-\cos(\theta°)} \qquad (100.1)$$

$$\text{Eqn. } (98.2) \Longrightarrow \mathrm{FQ} = \frac{2a}{1+\cos(\theta°)} \qquad (100.2)$$

$$\text{Eqn. } (98.3) \Longrightarrow \mathrm{FR} = \frac{2a}{1+\sin(\theta°)} \qquad (100.3)$$

$$\text{Eqn. } (98.4) \Longrightarrow \mathrm{FS} = \frac{2a}{1-\sin(\theta°)} \qquad (100.4)$$

Referring fig. 9,

$$\mathrm{PK} = \mathrm{FP} \times \sin(\theta°)$$

Substituting eqn. (100.1) in above,

$$PK = \frac{2a\sin(\theta°)}{1-\cos(\theta°)} \qquad (100.5)$$

Referring fig. 9,

$QM = FQ \times \sin(\theta°)$

Substituting eqn. (100.2) in above,

$$QM = \frac{2a\sin(\theta°)}{1+\cos(\theta°)} \qquad (100.6)$$

Referring fig. 9, If $\angle KFP = \theta°, \angle LFR = 90° - \theta°$

Referring fig. 9,

$RL = FR \times \sin(90 - \theta°)$

Substituting eqn. (100.3) in above,

$$\therefore RL = \frac{2a \times \sin(90-\theta°)}{1+\sin(\theta°)}$$

$$\therefore RL = \frac{2a\cos(\theta°)}{1+\sin(\theta°)} \qquad (100.7)$$

Referring fig. 9, If $\angle KFP = \theta°, \angle NSF = 90° - \theta°$

Referring fig. 9,

$SN = FS \times \sin(90 - \theta°)$

Substituting eqn. (100.4) in above,

$$\therefore SN = \frac{2a}{1-\sin(\theta°)} \times \sin(90-\theta°)$$

$$\therefore SN = \frac{2a\cos(\theta°)}{1-\sin(\theta°)} \qquad (100.8)$$

Multiplying eqns. (100.5) & (100.6),

$$PK \times QM = \left(\frac{2a\sin(\theta°)}{1-\cos(\theta°)}\right) \times \left(\frac{2a\sin(\theta°)}{1+\cos(\theta°)}\right)$$

$$\therefore \text{PK} \times \text{QM} = \frac{4a^2\sin^2(\theta°)}{1 - \cos^2(\theta°)}$$

$$\therefore \text{PK} \times \text{QM} = \frac{4a^2\sin^2(\theta°)}{\sin^2(\theta°)}$$

$$\therefore \text{PK} \times \text{QM} = 4 \times a^2$$

$$\therefore \text{PK} \times \text{QM} = (2a)^2$$

$$\therefore \text{PK} \times \text{QM} = (\text{FG})^2 \qquad (100.9)$$

Multiplying eqns. (100.7) & (100.8),

$$\text{RL} \times \text{SN} = \left(\frac{2a\cos(\theta°)}{1 + \sin(\theta°)}\right) \times \left(\frac{2a\cos(\theta°)}{1 - \sin(\theta°)}\right)$$

$$\therefore \text{RL} \times \text{SN} = \frac{4a^2\cos^2(\theta°)}{1 - \sin^2(\theta°)}$$

$$\therefore \text{RL} \times \text{SN} = \frac{4a^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

$$\therefore \text{RL} \times \text{SN} = 4 \times a^2$$

$$\therefore \text{RL} \times \text{SN} = (2a)^2$$

$$\therefore \text{RL} \times \text{SN} = (\text{FG})^2 \qquad (100.10)$$

where, FG is semi latus-rectum.

Equating eqns. (100.9) & (100.10),

$$\text{PK} \times \text{QM} = \text{RL} \times \text{SN} = (\text{FG})^2 \qquad (100.11)$$

Eqn. (100.11) is mathematical expression of the theorem.

CONCLUSION

This research has presented a comprehensive mathematical investigation into the properties of the parabola, with particular emphasis on its tangent, normal, and focus. Through the application of parametric equations, 100 new theorems have been systematically derived and rigorously proven, supported by illustrative diagrams to enhance clarity and comprehension. These results extend the classical understanding of the parabola, offering deeper insights into its geometric structure. Beyond pure mathematics, the findings carry significant implications for applied fields such as radar systems, antenna design, and optical devices like torch reflectors. The integration of parametric methods with detailed visualizations strengthens both the precision and accessibility of the work. Overall, this study provides a valuable reference for scholars and researchers engaged in advanced studies of conic sections, celestial mechanics, and geometric theory, marking a meaningful contribution to both mathematical knowledge and its interdisciplinary applications.

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