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Convex Property of Sandwich Theorem for Set Valued Functions

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Abstract: In this paper I generalize sandwich theorem for fuzzy set-valued functions with convex property which proved by E. Sodowska.

Keyword: Fuzzy set, set valued function, convex

1. Introduction

E.Sodowska proved a necessary and sufficient condition for the existence of a convex set-valued function H such that $F \subset H \subset G$ for any two given set-valued functions F and G. In this paper the authors present a necessary and sufficient condition for the existence of a convex fuzzy set-valued function H such that $\text{Supp } F_x \subset \text{Supp } H_x \subset \text{Supp } G_x$ for all $x \in C$, for any two given fuzzy set-valued functions F and G defined on a convex set $C \subset R$, where R denotes the field of real numbers.

If X denotes an ordinary non-empty set then we will denote the collection of all fuzzy sets by $\mathbf{I}^X$, where I denotes the closed unit interval [0,1].

2. Definitions

Definition 2.1 :(L.A. Zadeh [4]) Fuzzy Set

Let X be a space of points (objects), with a generic element of X denoted by x. Thus X={x}.

A fuzzy set (class) A in X is characterized by a membership (characteristic) function $f_A(x)$ which associates with each point in X a real number in the interval [0, 1], with the value of $f_A(x)$ at x representing the "grade of membership" of x in A. i.e.

A fuzzy set A on the domain X is defined as a mapping

$A: X \rightarrow [0, 1]$, where $[0, 1] = I$ is the range of A.

Definition 2.2: Fuzzy Space

The family of all fuzzy sets on X is I^X, consisting of all the mappings from X to I, I^X is called the fuzzy space $X \times I$ also represents the fuzzy space, in this case, the fuzzy set $A \subseteq X \times I, A \in I^X$ is called a crisp subset on X

Definition 2.3: Support A: Let A is a fuzzy set in X, then the support is defined as

$\text{Supp } A = \{x \in X : A(X) > 0\}$

Definition 2.4: Fuzzy Set valued: A function $F : X \to I^Y$ is said to be fuzzy set-valued from X into Y if F(x) is a fuzzy set in Y for each $x \in X$. In this paper, I will denote F(x) by F_x for convenience.

Definition 2.5: Fuzzy Set valued function: Let F be a fuzzy set-valued function from $D(\neq \phi)$ into X. By the graph Gr(F) of F, we mean the set

$$\{(x, u) \in D \times X : u \in \text{Supp } F_x\}.$$

Definition 2.6: Convex fuzzy set valued function: A fuzzy set-valued function F defined on a convex set $D \subset R$ is said to be convex if the following condition is satisfied:

If $F_x(u) > 0,\ F_y(v) > 0,\ 0 \le t \le 1$ then $F_{tx+(1-t)y}(tu + (1-t)v) > 0$

Definition 2.7: Graph of a Fuzzy Set-Valued Function
The graph of a fuzzy set-valued function is a fuzzy set Gr(F) on the Cartesian product space and its membership is defined as $Gr(F)(x, y) = F(x)(y)$

3. Theorems

Theorem 3.1: F is a convex fuzzy set-valued function if and only if Gr(F) is a convex set.

Proof: Follows from the definition of the graph of a fuzzy set-valued function is a fuzzy set Gr(F)

Theorem 3.2: Let D be a real interval. Let F, G be fuzzy set-valued functions defined on D into R, such that Gr(F) is the union of two connected subsets of R^2. Then F and G satisfy the condition:

If $F_x(u) > 0,\ F_y(v) > 0,\ 0 \le t \le 1$ then $G_{tx+(1-t)y}(tu + (1-t)v) > 0$

If there exists a convex fuzzy set-valued function $H : D \to I^R$ such that

$$\text{Supp } F_x \subset \text{Supp } H_x \subset \text{Supp } G_x \text{ for all } x \in D$$

Proof: We assume F and G satisfy the given conditon. Define a fuzzy set valued function

$H : D \to I^R$ as follows:

$H_x(z) = F_x(z)$ if $(x, z) \in \text{conv}(Gr(F))$

And $H_x(z) = 0$ otherwise.

Then we have $\text{Supp } F_x \subset \text{Supp } H_x$ for all $x \in D$

Also Gr(H) = conv(Gr(F)) is a convex subset of R^2

Thus H is a convex fuzzy set-valued function.

We show $\operatorname{Supp} H_x \subset \operatorname{Supp} G_x$

Let $x \in D$ and $u \in \operatorname{Supp} H_x$ Then $(x, u) \in \operatorname{conv}(Gr(F))$ Since Gr(F) is the union of two connected subsets of R^2, each element of its convex hull is a convex combination of two elements of Gr(F). That is, there exist

$(x_1, u_1), (x_2, u_2) \in Gr(F)$and $t \in [0, 1]$ such that

$(x, u) = t(x_1, u_1) + (1 - t)(x_2, u_2)$

That is, $x = tx_1 + (1 - t)x_2$ and $u = tu_1 + (1 - t)u_2$

This means $F_{x_1}(u_1) > 0$, $F_{x_2}(u_2) > 0$, and $t \in [0, 1]$

Hence $G_{tx_1+(1-t)x_2}(tu_1 + (1 - t)u_2) > 0$

$G_x(u) > 0$ i.e. $u \in \operatorname{Supp} G_x$

Hence $\operatorname{Supp} F_x \subset \operatorname{Supp} H_x \subset \operatorname{Supp} G_x$

The converse is easy and requires no connectedness condition.

4. Corollary

Let I be a real interval and $F, G: I \to n(R)$ be given set-valued functions such that Gr(F) is a union of two connected subsets ofR^2.Then F and G satisfy the condition:

$F(x) + (1 - t)F(y) \subset G(tx + (1 - t)y), \ x, y \in D, \ t \in [0, 1]$

If and only if there exist a convex set-valued function $H: I \to n(R)$ such that

$F(x) \subset H(x) \subset G(x)$for all $x \in I$

5. Conclusion

The condition

$F_x(u) > 0, \ F_y(v) > 0, \ 0 \le t \le 1$ then $G_{tx+(1-t)y}(tu + (1 - t)v) > 0$

Is a generalization of the condition

$F(x) + (1 - t)F(y) \subset G(tx + (1 - t)y), \ x, y \in D, \ t \in [0, 1]$

and is a generalization of set valued function to fuzzy set valued function.

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