

Soft Fixed Point Theorems with Soft Altering Distance Functions

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Abstract: In this study, we develop new soft fixed point theorems that extend the soft Banach contraction principle in the setting of soft S-metric spaces. These results are formulated by employing one or more soft altering distance functions applied to various types of soft contraction mappings. Furthermore, we establish several additional results and corollaries within the framework of soft S-metric spaces.

Keywords: Soft fixed point, Soft contractive conditions, Soft S-metric space, Soft altering distance function.

MSC: 47H10, 54H25

1. Introduction

Fixed point theory, pioneered by Brouwer in 1912, has had profound influence due to its applications in diverse areas such as optimization theory, differential equations, variational inequalities, complementary problems, equilibrium theory, game theory, and economics. The field was further advanced in 1922 when Banach [2] established the contraction principle. Building on these foundations, many scholars have studied different contraction and contractive mappings to investigate fixed point and common fixed point results within metric spaces and generalized metric spaces. which can be explored in ([7]-[13]).

In 1984, Sessa [21], M. S. Khan, and M. Swalech [13] expanded the scope of metric fixed point theory by introducing a control function, known as an altering distance function. More recently, J. R. Morales and E. M. Rojas [20] derived fixed point theorems using altering distance functions through rational expressions, while Manish Sharma and A. S. Saluja [17] also developed fixed point theorems based on altering distance functions.

The concept of a soft set was introduced by Molodtsov [19] in 1999 as a new mathematical tool for addressing uncertainty. A soft set can be viewed as a parameterized collection that provides approximate descriptions of objects. Research in soft set theory progressed rapidly after Maji *et al.* [15,16] defined several operations on soft sets. Over the years, soft set theory has continued to develop and has shown promising applications in various fields. Das and Samanta [5,6] introduced the concepts of soft elements, soft real numbers, and soft points, along with their key properties. Building on these

foundations, they proposed the notion of a soft metric [6]. Wardowski [22] later defined soft mappings and established several fixed point results. Moreover, Abbas *et al.* [1] introduced soft contraction mappings based on the theory of soft elements in soft metric spaces and investigated their fixed points, deriving several significant results. Cigdem Gunduz Aras *et al.* [3,4] proposed the concept of soft S-metric spaces and examined their key properties. They further established corresponding soft fixed point theorems.

In the next section we focus on the key definitions and properties needed to understand the results presented in this paper. Additional details and explanations can be found in the cited references.

2. Preliminaries

Definition 2.1.[19]: “A pair (F, E) is called a soft set over a given universal set X , if and only if F is a mapping from a set of parameters E (each parameter could be a word or a sentence) into the power set of X denoted by $P(X)$. That is, $F: E \rightarrow P(X)$. Clearly, a soft set over X is a parameterized family of subsets of the given universe X .”

Example 2.2.[19] “Suppose a person wants to buy a television. So let X denotes the number of televisions which he saw in different showrooms as $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$. Let E is the set of parameters where $E = \{e_1, e_2, e_3, e_4, e_5\} = \{\text{android, branded, 55 inches, full HD, refresh rate}\}$. Suppose that $F(e_1) = \{h_1, h_2, h_4, h_5\}$, $F(e_2) = \{h_1, h_3, h_5\}$, $F(e_3) = \{h_1, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$, $F(e_5) = \{h_1, h_5, h_6\}$, then the soft set (F, E) can be looked as a collection of approximations as below :

$(F, E) = \{\text{android} = \{h_1, h_2, h_4, h_5\}, \text{branded} = \{h_1, h_3, h_5\}, 55 \text{ inches} = \{h_1, h_5\},$

$\text{full HD} = \{h_1, h_3, h_5\}, \text{refresh rate} = \{h_1, h_5, h_6\}\}.$ ”

Definition 2.3.[16]: “A soft set (F, E) over X is said to be a null soft set denoted by $\tilde{\Phi}$, if for all $e \in E, F(e) = \text{null set } \phi$.”

Definition 2.4.[16]: “A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$ if for all $e \in E, F(e) = X$.”

Definition 2.5.[5]: “Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F: E \rightarrow \mathcal{B}(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ etc $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0, \tilde{1}(e) = 1$, for all $e \in E$ respectively.”

Definition 2.6.[5]: “Let $(\tilde{X}, S, E)$ be a soft S-metric space. A map $(T, \varphi): (\tilde{X}, S, E) \rightarrow (\tilde{X}, S, E)$ is said to be a soft contraction mapping if there exists a soft real number $\bar{k} \in \mathbb{R}(E)$, $\tilde{0} \leq \bar{k} < \tilde{1}$ (where $\mathbb{R}(E)$ denotes the soft real number set) such that

$$S((T, \varphi)(\hat{u}_a), (T, \varphi)(\hat{u}_a), (T, \varphi)(\hat{v}_b)) \leq \bar{k} S(\hat{u}_a, \hat{u}_a, \hat{v}_b),$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$.”

In 2018, Aras *et al.* [3] introduced the concept of soft S-metric spaces and also discussed its important properties which are as follows:

“Let $\tilde{X}$ be an absolute soft set, E be a non-empty set of parameters and $SP(\tilde{X})$ be the collection of all soft points of $\tilde{X}$. Let $\mathbb{R}(E)^*$ denotes the set of all non-negative soft real numbers.”

Definition 2.7.[3] “A soft S-metric on $\tilde{X}$ is a mapping $\mathcal{S} : SP(\tilde{X}) \times SP(\tilde{X}) \times SP(\tilde{X}) \rightarrow \mathbb{R}(E)^*$ which satisfies the following conditions:

$$(\overline{\mathcal{S}}_1) \mathcal{S}(\hat{u}_a, \hat{v}_b, \hat{w}_c) \geq \tilde{0};$$

$$(\overline{\mathcal{S}}_2) \mathcal{S}(\hat{u}_a, \hat{v}_b, \hat{w}_c) = \tilde{0}, \text{ if and only if } \hat{u}_a = \hat{v}_b = \hat{w}_c;$$

$$(\overline{\mathcal{S}}_3) \mathcal{S}(\hat{u}_a, \hat{v}_b, \hat{w}_c) \leq \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{t}_d) + \mathcal{S}(\hat{v}_b, \hat{v}_b, \hat{t}_d) + \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d).$$

For all $\hat{u}_a, \hat{v}_b, \hat{w}_c, \hat{t}_d \in SP(\tilde{X})$, then the soft set $\tilde{X}$ with a soft S-metric is called soft S-metric space and denoted by $(\tilde{X}, \mathcal{S}, E)$.”

Lemma 2.8.[3] “Let $(\tilde{X}, \mathcal{S}, E)$ is a soft S-metric space. Then we have

$$\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b) = \mathcal{S}(\hat{v}_b, \hat{v}_b, \hat{u}_a).”$$

Definition 2.9.[3] “A soft sequence $\{\hat{u}_{a_n}^n\}$ in $(\tilde{X}, \mathcal{S}, E)$ converges to $\hat{v}_b$ if and only if $\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{v}_b) \rightarrow \tilde{0}$ as $n \rightarrow \infty$ and we denote this by $\lim_{n \rightarrow \infty} \hat{u}_{a_n}^n = \hat{v}_b$.”

Definition 2.10.[3] “A soft sequence $\{\hat{u}_{a_n}^n\}$ in $(\tilde{X}, \mathcal{S}, E)$ is called a Cauchy sequence if for $\tilde{\varepsilon} > \tilde{0}$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_m}^m) < \tilde{\varepsilon}$ for each $m, n \geq n_0$.”

Definition 2.11.[3] “A soft S-metric space $(\tilde{X}, \mathcal{S}, E)$ is said to be complete if every Cauchy sequence is convergent.”

Definition 2.12.[4] “Let $(\tilde{X}, \mathcal{S}, E)$ and $(\tilde{Y}, \mathcal{S}', E')$ be two soft S-metric spaces. The mapping $f_\varphi : (\tilde{X}, \mathcal{S}, E) \rightarrow (\tilde{Y}, \mathcal{S}', E')$ is a soft mapping, where $f : \tilde{X} \rightarrow \tilde{Y}$ and $\varphi : E \rightarrow E'$ are two mappings.”

Definition 2.13.[4] “Let $f_\varphi : (\tilde{X}, \mathcal{S}, E) \rightarrow (\tilde{Y}, \mathcal{S}', E')$ be a soft mapping from soft S-metric space $(\tilde{X}, \mathcal{S}, E)$ to a soft S-metric space $(\tilde{Y}, \mathcal{S}', E')$. Then f_φ is soft continuous at a soft point $\hat{u}_a \in SP(\tilde{X})$ if and only if $f_\varphi(\{\hat{u}_{a_n}^n\}) \rightarrow f_\varphi(\hat{u}_a)$.”

Definition 2.14.[4] “Let $(\tilde{X}, \mathcal{S}, E)$ be a soft S-metric space. A map $f_\varphi : (\tilde{X}, \mathcal{S}, E) \rightarrow (\tilde{X}, \mathcal{S}, E)$ is said to be a soft contraction mapping if there exists a soft real number $\tilde{k} \in \mathbb{R}(E)$, $\tilde{0} \leq \tilde{k} < \tilde{1}$ (where $\mathbb{R}(E)$ denotes the soft real number set) such that

$$\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) \leq \tilde{k} \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b),$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$.”

Definition 2.15.[13] “A function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called an altering distance function if the following property is satisfied:

$$(\Theta_1) \psi(0) = 0,$$

(Θ_2) ψ is monotonically non-decreasing function,

(Θ_3) ψ is a continuous function,

By Ψ we denote the set of all altering distance functions.”

In 2018, Elif G. *et al.* [9] establish the following definition of soft altering distance function in soft metric space.

Definition 2.16.[9] “A soft function $\psi : \mathbb{R}(E)^* \rightarrow \mathbb{R}(E)^*$ is called a soft altering distance function if ψ satisfies the following property:

$$(\Theta_1) \psi(\bar{0}) = \bar{0},$$

(Θ_2) ψ is monotonically non-decreasing function,

(Θ_3) ψ is a sequentially continuous function i.e., $\hat{u}_{a_n}^n \rightarrow \hat{u}_a$, then $\psi(\hat{u}_{a_n}^n) \rightarrow \psi(\hat{u}_a)$.”

Theorem 2.17.[9] “Let $(\tilde{X}, d)$ be a complete metric space. Let $\psi : \mathbb{R}(E)^* \rightarrow \mathbb{R}(E)^*$ be a soft altering distance function and $T : \tilde{X} \rightarrow \tilde{X}$ be a soft mapping which satisfies the following inequality:

$$\psi(d(T(\hat{u}_{a_n}^n), T(\hat{v}_{b_n}^n))) \leq \bar{c} \psi(d(\hat{u}_a, \hat{v}_b)),$$

for some $\bar{0} < \bar{c} < \bar{1}$ and $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$. Then T has a unique soft fixed point.”

3. Main Result

Theorem 3.1: Let $(\tilde{X}, \mathcal{S}, E)$ be a complete soft S-metric space. Let $\psi : \mathbb{R}(E)^* \rightarrow \mathbb{R}(E)^*$ be a soft altering distance function with $\psi(\bar{x}) \neq \bar{0}$ for all $\bar{x} \neq \bar{0}$ and f_φ be soft self mapping on $(\tilde{X}, \mathcal{S}, E)$ which satisfies the following inequality:

$$\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) \leq \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b) - \psi(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)), \quad (3.1)$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$. Then f_φ has a unique soft fixed point in $\tilde{X}$.

Proof: Let $\hat{u}_{a_0}^0 \in SP(\tilde{X})$ be an arbitrary point and let $\{\hat{u}_{a_n}^n\}$ be a soft sequence defined as follows

$$\hat{u}_{a_{n+1}}^{n+1} = f_\varphi(\hat{u}_{a_n}^n) = f_\varphi^{n+1}(\hat{u}_a), \hat{t}_n = \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

We first prove that f_φ has a soft fixed point in $(\tilde{X}, \mathcal{S}, E)$. We may assume that $\hat{t}_n > \bar{0}$ for each $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.1), we obtain

$$\begin{aligned} \mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_{n+1}}^{n+1})) &\leq \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) - \psi(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})) \\ \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+2}}^{n+2}) &\leq \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) - \psi(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})) \\ \hat{t}_{n+1} &\leq \hat{t}_n - \psi(\hat{t}_n) \leq \hat{t}_n. \end{aligned} \quad (3.2)$$

Since $\{\hat{t}_n\}$ is a decreasing sequence of soft real numbers. Hence $\{\hat{t}_n\}$ has a limit point. We put $\lim_{n \rightarrow \infty} \hat{t}_n = \tilde{t}$ and suppose $\tilde{t} > \bar{0}$. Since ψ is non decreasing, $\hat{t}_n \geq \tilde{t}$ implies that $\psi(\hat{t}_n) \geq \psi(\tilde{t}) > \bar{0}$. By (3.2) we have $\hat{t}_{n+1} \leq \hat{t}_n - \psi(\tilde{t})$.

Thus, $\hat{t}_{n+M} \leq \hat{t}_n - M \psi(\tilde{t})$ is a contradiction for M large enough. So $\tilde{t} = \bar{0}$. Therefore, $\{\hat{t}_n\}$ converges to $\bar{0}$. As in above theorem it is easy to show that $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence in $(\tilde{\mathcal{X}}, \mathcal{S}, E)$. By completeness of $(\tilde{\mathcal{X}}, \mathcal{S}, E)$, $\{\hat{u}_{a_n}^n\}$ converges to some soft point $\hat{w}_c$.

Now, we show that $\hat{w}_c$ is a fixed soft point of f_φ . If we substitute $\hat{u}_a = \hat{u}_{a_{n-1}}^{n-1}$ and $\hat{v}_b = \hat{w}_c$ in (3.1), we obtain

$$\begin{aligned} \mathcal{S}(f_\varphi(\hat{u}_{a_{n-1}}^{n-1}), f_\varphi(\hat{u}_{a_{n-1}}^{n-1}), f_\varphi(\hat{w}_c)) &\leq \mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c) - \psi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)) \\ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, f_\varphi(\hat{w}_c)) &\leq \mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c) - \psi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)). \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using the continuity of ψ and f_φ , we get

$$\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c)) \leq \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{w}_c) - \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{w}_c)) = \psi(\bar{0}) = \bar{0},$$

which implies $\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c)) = \bar{0}$ that is $f_\varphi(\hat{w}_c) = \hat{w}_c$.

To prove the uniqueness, we assume that $\hat{w}_c$ and $\hat{r}_d$ be two different fixed soft point of f_φ . Then from (3.1), we obtain that

$$\begin{aligned} \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), f_\varphi(\hat{r}_d)) &\leq \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d) - \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) \\ \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d) &\leq \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d) - \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)). \end{aligned}$$

Which implies $\psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) \leq \bar{0}$.

Thus, $\psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) = \bar{0}$ and hence we get $\hat{w}_c = \hat{r}_d$.

Therefore, f_φ has a soft unique fixed point.

Here completes the proof.

Note: If we consider $\psi(\bar{t}) = \bar{k} \cdot \bar{t}$ where $\bar{0} < \bar{k} \leq \bar{1}$, then the above theorem reduces to contraction condition

$$\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) \leq \bar{c} \mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b),$$

for some $\bar{0} < \bar{c} < \bar{1}$, which is given by Aras [4].

Theorem 3.2: Let $(\tilde{\mathcal{X}}, \mathcal{S}, E)$ be a complete soft S-metric space. Let $\psi, \phi : \mathbb{R}(E)^* \rightarrow \mathbb{R}(E)^*$ be the two soft altering distance function with $\psi(\tilde{x}) \neq \bar{0}$ and $\phi(\tilde{x}) \neq \bar{0}$ for all $\tilde{x} \neq \bar{0}$ and f_φ be soft self mapping on $(\tilde{\mathcal{X}}, \mathcal{S}, E)$ which satisfies the following inequality:

$$\psi(\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b))) \leq \psi(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)) - \phi(\mathcal{S}(\hat{u}_a, \hat{u}_a, \hat{v}_b)), \quad (3.3)$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$. Then f_φ has a unique soft fixed point in $\tilde{X}$.

Proof: Let $\hat{u}_{a_0}^0 \in SP(\tilde{X})$ and let $\{\hat{u}_{a_n}^n\}$ be a soft sequence defined as follows

$$\hat{u}_{a_{n+1}}^{n+1} = f_\varphi(\hat{u}_{a_n}^n) = f_\varphi^{n+1}(\hat{u}_a), \hat{t}_n = \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

We first prove that f_φ has a soft fixed point in $(\tilde{X}, \mathcal{S}, E)$. We may assume that $\hat{t}_n > \bar{0}$ for each $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.3), we obtain

$$\begin{aligned} \psi\left(\mathcal{S}\left(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_{n+1}}^{n+1})\right)\right) &\leq \psi\left(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})\right) - \phi\left(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})\right) \\ \psi\left(\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+2}}^{n+2})\right) &\leq \psi\left(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})\right) - \phi\left(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})\right) \\ \psi(\hat{t}_{n+1}) &\leq \psi(\hat{t}_n) - \phi(\hat{t}_n) \leq \psi(\hat{t}_n). \end{aligned} \quad (3.4)$$

Since ψ is non-decreasing function, $\{\hat{t}_n\}$ is a decreasing sequence of soft real numbers. Hence $\{\hat{t}_n\}$ has a limit point. We put $\lim_{n \rightarrow \infty} \hat{t}_n = \tilde{t}$ and suppose $\tilde{t} > \bar{0}$. Letting $n \rightarrow \infty$ in (3.4) and using continuity of ψ , we obtain $\psi(\hat{t}_n) \leq \psi(\tilde{t}) - \phi(\tilde{t}) < \psi(\tilde{t})$ which is a contradiction. So $\tilde{t} = \bar{0}$.

Therefore, $\{\hat{t}_n\}$ converges to $\bar{0}$.

Now, we will prove that $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence in $(\tilde{X}, \mathcal{S}, E)$. Suppose that $\{\hat{u}_{a_n}^n\}$ is not a Cauchy sequence which means that there is a constant $\bar{\epsilon} > \bar{0}$ and two subsequence $\{\hat{u}_{a_{n_k}}^{n_k}\}$ and $\{\hat{u}_{a_{m_k}}^{m_k}\}$ of $\{\hat{u}_{a_n}^n\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, we find that $n_k > m_k > n$, $\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) \geq \bar{\epsilon}$ and $\mathcal{S}(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k}) < \bar{\epsilon}$. For each $n > 0$, we put

$$\begin{aligned} \widetilde{B}_n &= \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}). \text{ Then we have} \\ \bar{\epsilon} &\leq \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) \leq 2\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k-1}}^{n_k-1}) + \mathcal{S}(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k}) \\ &\leq 2\hat{t}_{n-1} + \bar{\epsilon} \end{aligned}$$

Since $\{\hat{t}_n\}$ converges to $\bar{0}$, we obtain $\{\widetilde{B}_n\}$ converges to $\bar{\epsilon}$.

Similarly, we can show that $\mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1})$ converges to $\bar{\epsilon}$.

From the hypothesis, we deduce

$$\begin{aligned} \psi\left(\mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}))\right) &\leq \psi\left(\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k})\right) - \phi\left(\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k})\right) \\ \psi\left(\mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1})\right) &\leq \psi\left(\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k})\right) - \phi\left(\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k})\right) \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain that $\psi(\bar{\epsilon}) \leq \psi(\bar{\epsilon}) - \phi(\bar{\epsilon}) \leq \psi(\bar{\epsilon})$, which is contradiction. Hence, $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence. By completeness of $(\tilde{X}, \mathcal{S}, E)$, $\{\hat{u}_{a_n}^n\}$ converges to some soft point $\hat{w}_c$.

Now, we show that $\hat{w}_c$ is a fixed soft point of f_φ . If we substitute $\hat{u}_a = \hat{u}_{a_{n-1}}^{n-1}$ and $\hat{v}_b = \hat{w}_c$ in (3.3), we obtain

$$\begin{aligned}\psi(\mathcal{S}(f_\varphi(\hat{u}_{a_{n-1}}^{n-1}), f_\varphi(\hat{u}_{a_{n-1}}^{n-1}), f_\varphi(\hat{w}_c))) &\leq \psi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)) - \phi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)) \\ \psi(\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, f_\varphi(\hat{w}_c))) &\leq \psi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)) - \phi(\mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{w}_c)).\end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using the continuity of ψ and f_φ , we get

$$\psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c))) \leq \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{w}_c)) - \phi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{w}_c)) = \bar{0},$$

which implies $\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c)) = \bar{0}$ that is $f_\varphi(\hat{w}_c) = \hat{w}_c$.

To prove the uniqueness, we assume that $\hat{w}_c$ and $\hat{r}_d$ be two different fixed soft point of f_φ . Then from (3.3), we obtain that

$$\begin{aligned}\psi(\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), f_\varphi(\hat{r}_d))) &\leq \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) - \phi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) \\ \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) &\leq \psi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) - \phi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)).\end{aligned}$$

Which implies $\phi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) \leq \bar{0}$. Thus, $\phi(\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{r}_d)) = \bar{0}$ and hence we get $\hat{w}_c = \hat{r}_d$.

Therefore, f_φ has a soft unique fixed point.

Note: In Theorem 3.2, if we particularly take $\phi(\bar{t}) = (\bar{1} - \bar{k}) \psi(\bar{t})$, for all $\bar{t} > \bar{0}$ where $\bar{0} < \bar{k} < \bar{1}$ then we obtain the result of Aras]. Again, by taking $\psi(\bar{t}) = \bar{t}$ for all $\bar{t} > \bar{0}$, in Theorem 3.2, we obtain the result of Theorem 3.1.

Theorem 3.3: Let $(\tilde{\mathcal{X}}, \mathcal{S}, E)$ be a complete soft S-metric space. Let $\psi \in \Psi$ and let $f_\varphi: (\tilde{\mathcal{X}}, \mathcal{S}, E) \rightarrow (\tilde{\mathcal{X}}, \mathcal{S}, E)$ be a soft self mapping which satisfies the following condition:

$$\begin{aligned}\psi\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b))\} &\leq \alpha \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{u}_a), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\} \\ &\quad + \beta \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b)\} \\ &\quad + \gamma \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\},\end{aligned}\tag{3.5}$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{\mathcal{X}})$ with $\hat{u}_a \neq \hat{v}_b$, and for some $\alpha, \beta, \gamma > \bar{0}$ with $3\alpha + \beta + 3\gamma < \bar{1}$ then, f_φ has a unique soft fixed point $\hat{w}_c \in SP(\tilde{\mathcal{X}})$ and moreover for each soft point $\hat{u}_a$ we have $\lim_{n \rightarrow \infty} f_\varphi^n \hat{u}_a = \hat{w}_c$.

Proof: Let $\hat{u}_{a_0}^0 \in SP(\tilde{\mathcal{X}})$ be an arbitrary point and $\{\hat{u}_{a_n}^n\}$ be a sequence defined as follows

$$\hat{u}_{a_{n+1}}^{n+1} = f_\varphi(\hat{u}_{a_n}^n) = f_\varphi^{n+1}(\hat{u}_{a_0}^0), \text{ for each } n > 0. \text{ Then from (3.5) we obtain}$$

$$\psi\{\mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1})\}$$

$$\begin{aligned}
 &= \psi \left\{ \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), f_{\varphi}(\hat{u}_{a_n}^n) \right) \right\} \\
 &\leq \alpha \psi \max \left\{ \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), \hat{u}_{a_{n-1}}^{n-1} \right), \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_n}^n), f_{\varphi}(\hat{u}_{a_n}^n), \hat{u}_{a_{n-1}}^{n-1} \right) \right\} \\
 &+ \beta \psi \max \left\{ \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), \hat{u}_{a_n}^n \right), \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_n}^n), f_{\varphi}(\hat{u}_{a_n}^n), \hat{u}_{a_n}^n \right) \right\} \\
 &+ \gamma \psi \max \left\{ \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), f_{\varphi}(\hat{u}_{a_{n-1}}^{n-1}), \hat{u}_{a_n}^n \right), \mathcal{S} \left(f_{\varphi}(\hat{u}_{a_n}^n), f_{\varphi}(\hat{u}_{a_n}^n), \hat{u}_{a_{n-1}}^{n-1} \right) \right\} \\
 &\leq \alpha \psi \max \left\{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n-1}}^{n-1}), \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n-1}}^{n-1}) \right\} \\
 &+ \beta \psi \max \left\{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_n}^n), \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n) \right\} \\
 &+ \gamma \psi \max \left\{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_n}^n), \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n-1}}^{n-1}) \right\} \\
 &\leq \alpha \psi \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n-1}}^{n-1}) + \beta \psi \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n) + \gamma \psi \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n-1}}^{n-1}) \\
 &\leq \alpha \psi \{ 2\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n) + \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n-1}}^{n-1}) \} + \beta \psi \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n) \\
 &+ \gamma \psi \{ 2\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n) + \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n-1}}^{n-1}) \},
 \end{aligned}$$

which implies that

$$\psi \{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \} \leq \left(\frac{\alpha + \gamma}{1 - 2\alpha - \beta - 2\gamma} \right) \mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_n}^n)$$

Thus, we have

$$\begin{aligned}
 \psi \{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \} &\leq \tilde{k} \psi \{ \mathcal{S}(\hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_{n-1}}^{n-1}, \hat{u}_{a_n}^n) \} \text{ where } \tilde{k} = \frac{\alpha + \gamma}{1 - 2\alpha - \beta - 2\gamma} \\
 &\leq \tilde{k}^2 \psi \{ \mathcal{S}(\hat{u}_{a_{n-2}}^{n-2}, \hat{u}_{a_{n-2}}^{n-2}, \hat{u}_{a_{n-1}}^{n-1}) \} \\
 &\dots\dots\dots \\
 &\dots\dots\dots
 \end{aligned}$$

$$\psi \{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \} \leq \tilde{k}^n \psi \{ \mathcal{S}(\hat{u}_{a_0}^0, \hat{u}_{a_0}^0, \hat{u}_{a_1}^1) \} \quad (3.6)$$

Since $0 \leq \tilde{k} < 1$, from (4.3.9) we obtain $\lim_{n \rightarrow \infty} \psi \{ \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) \} = 0$.

$$\text{From the fact that } \psi \in \Psi, \text{ we have } \lim_{n \rightarrow \infty} \mathcal{S}(\hat{u}_{a_n}^n, \hat{u}_{a_n}^n, \hat{u}_{a_{n+1}}^{n+1}) = 0. \quad (3.7)$$

Now, we will prove that $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence in $(\tilde{\mathcal{X}}, \mathcal{S}, E)$. Suppose that $\{\hat{u}_{a_n}^n\}$ is not a Cauchy sequence which means that there is a constant $\bar{\epsilon}_0 > 0$ and two subsequence $\{\hat{u}_{a_{n_k}}^{n_k}\}$ and $\{\hat{u}_{a_{m_k}}^{m_k}\}$ of $\{\hat{u}_{a_n}^n\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, we find that $n_k > m_k > n$, $\mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) \geq \bar{\epsilon}_0$ and $\mathcal{S}(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k}) < \bar{\epsilon}_0$. For each $n > 0$, Then we have

$$\bar{\epsilon}_0 \leq \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) \leq 2 \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k-1}}^{n_k-1}) + \mathcal{S}(\hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{n_k-1}}^{n_k-1}, \hat{u}_{a_{m_k}}^{m_k})$$

Taking limit as $n \rightarrow \infty$, from (3.7) we obtain

$$\bar{\epsilon}_0 \leq \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) < \bar{\epsilon}_0,$$

which implies that

$$\lim_{n \rightarrow \infty} \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k}}^{m_k}) = \bar{\epsilon}_0 \quad (3.8)$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}) = \bar{\epsilon}_0 \text{ and } \lim_{n \rightarrow \infty} \mathcal{S}(\hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{n_k}}^{n_k}, \hat{u}_{a_{m_k+1}}^{m_k+1}) = \bar{\epsilon}_0 \quad (3.9)$$

From the hypothesis, we deduce

$$\begin{aligned} & \psi(\mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}))) \\ & \leq \alpha \psi \max \{ \mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), \hat{u}_{a_{n_k}}^{n_k}), \mathcal{S}(f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), \hat{u}_{a_{n_k}}^{n_k}) \} \\ & + \beta \psi \max \{ \mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), \hat{u}_{a_{m_k}}^{m_k}), \mathcal{S}(f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), \hat{u}_{a_{m_k}}^{m_k}) \} \\ & + \gamma \psi \max \{ \mathcal{S}(f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), f_\varphi(\hat{u}_{a_{n_k}}^{n_k}), \hat{u}_{a_{m_k}}^{m_k}), \mathcal{S}(f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), f_\varphi(\hat{u}_{a_{m_k}}^{m_k}), \hat{u}_{a_{n_k}}^{n_k}) \} \\ & \leq \alpha \psi \max \{ \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k}}^{n_k}), \mathcal{S}(\hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{n_k}}^{n_k}) \} \\ & + \beta \psi \max \{ \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k}}^{m_k}), \mathcal{S}(\hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k}}^{m_k}) \} \\ & + \gamma \psi \max \{ \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k}}^{m_k}), \mathcal{S}(\hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{n_k}}^{n_k}) \} \end{aligned}$$

Using (3.7), (3.8) and (3.9) we obtain

$$\begin{aligned} \psi(\epsilon_0) &= \lim_{n \rightarrow \infty} \psi(\mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1})) \\ &\leq \alpha \lim_{n \rightarrow \infty} \psi \{ \mathcal{S}(\hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{n_k}}^{n_k}) \} + \beta \lim_{n \rightarrow \infty} \psi \{ \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k}}^{m_k}) \} \\ &+ \gamma \lim_{n \rightarrow \infty} \psi \{ \max \{ \mathcal{S}(\hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{n_k+1}}^{n_k+1}, \hat{u}_{a_{m_k}}^{m_k}), \mathcal{S}(\hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{m_k+1}}^{m_k+1}, \hat{u}_{a_{n_k}}^{n_k}) \} \} \\ &\leq (\alpha + \beta + \gamma) \psi(\epsilon_0). \end{aligned}$$

Since ϵ_0 is arbitrary, we get

$$\psi(\epsilon_0) \leq (\gamma + \beta + \delta) \psi(\epsilon_0),$$

as $(\gamma + \beta + \delta) \in (\bar{0}, \bar{1})$, thus we get a contradiction, then $\{\hat{u}_{a_n}^n\}$ is a Cauchy sequence in a complete metric space $(\tilde{\mathcal{X}}, \mathcal{S}, E)$. Thus there exist a soft point $\hat{w}_c$ such that $\lim_{n \rightarrow \infty} \hat{u}_{a_n}^n = \hat{w}_c$.

Again taking $\hat{u}_a = \hat{u}_{a_n}^n$ and $\hat{v}_b = \hat{w}_c$ in (3.5) we get

$$\begin{aligned} & \psi \{ \mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, f_\varphi(\hat{w}_c)) \} \\ &= \psi \{ \mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{w}_c)) \} \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), \hat{u}_{a_n}^n), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{u}_{a_n}^n)\} \\
 &+ \beta \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), \hat{w}_c), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\} \\
 &+ \gamma \psi \max\{\mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), \hat{w}_c), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{u}_{a_n}^n)\} \\
 &\leq \alpha \psi \max\{\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_n}^n), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{u}_{a_n}^n)\} \\
 &+ \beta \psi \max\{\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{w}_c), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\} \\
 &+ \gamma \psi \max\{\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, \hat{w}_c), \mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{u}_{a_n}^n)\}, \\
 &\lim_{n \rightarrow \infty} \psi \{\mathcal{S}(\hat{u}_{a_{n+1}}^{n+1}, \hat{u}_{a_{n+1}}^{n+1}, f_\varphi(\hat{w}_c))\} = \lim_{n \rightarrow \infty} \psi \mathcal{S}(f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{u}_{a_n}^n), f_\varphi(\hat{w}_c)) \leq \\
 &\alpha \psi \{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\} + \beta \psi \{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\} \\
 &\quad + \gamma \psi \{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\}.
 \end{aligned}$$

$$\Rightarrow \psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c))\} \leq (\alpha + \beta + \gamma) \psi \{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c)\}.$$

Since $\alpha + \beta + \gamma < \bar{1}$, then $\psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c))\} = 0 \Rightarrow \mathcal{S}(\hat{w}_c, \hat{w}_c, f_\varphi(\hat{w}_c)) = 0$.

Thus, $f_\varphi(\hat{w}_c) = \hat{w}_c$. Therefore, $\hat{w}_c$ is a fixed soft point of f_φ .

Now we are going to establish the uniqueness of soft fixed point.

For that let us suppose that $\hat{w}_c$ and $\hat{t}_d$ be two soft fixed point of f_φ with $\hat{w}_c \neq \hat{t}_d$.

Taking $\hat{u}_a = \hat{w}_c$ and $\hat{v}_b = \hat{t}_d$ in (3.5) we get

$$\begin{aligned}
 \psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d)\} &= \psi \left\{ \mathcal{S} \left(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), f_\varphi(\hat{t}_d) \right) \right\} \\
 &\leq \alpha \psi \max\{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{w}_c), \mathcal{S}(f_\varphi(\hat{t}_d), f_\varphi(\hat{t}_d), \hat{w}_c)\} \\
 &\quad + \beta \psi \max\{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{t}_d), \mathcal{S}(f_\varphi(\hat{t}_d), f_\varphi(\hat{t}_d), \hat{t}_d)\} \\
 &\quad + \gamma \psi \max\{\mathcal{S}(f_\varphi(\hat{w}_c), f_\varphi(\hat{w}_c), \hat{t}_d), \mathcal{S}(f_\varphi(\hat{t}_d), f_\varphi(\hat{t}_d), \hat{w}_c)\} \\
 &\leq \alpha \psi \max\{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{w}_c), \mathcal{S}(\hat{t}_d, \hat{t}_d, \hat{w}_c)\} \\
 &\quad + \beta \psi \max\{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d), \mathcal{S}(\hat{t}_d, \hat{t}_d, \hat{t}_d)\} \\
 &\quad + \gamma \psi \max\{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d), \mathcal{S}(\hat{t}_d, \hat{t}_d, \hat{w}_c)\}. \\
 \psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d)\} &\leq (\alpha + \beta + \gamma) \psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d)\}, \tag{3.10}
 \end{aligned}$$

as $\alpha + \beta + \gamma < \bar{1}$, thus from (3.10), we get a contradiction and hence

$\psi \{\mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d)\} = 0 \Rightarrow \mathcal{S}(\hat{w}_c, \hat{w}_c, \hat{t}_d) = 0$ which further implies that $\hat{w}_c = \hat{t}_d$.

Therefore, the fixed soft point we get is unique.

Here completes the proof.

Corollary 3.4: Let $(\tilde{X}, \mathcal{S}, E)$ be a complete soft S-metric space and let $f_\varphi: (\tilde{X}, \mathcal{S}, E) \rightarrow (\tilde{X}, \mathcal{S}, E)$ be a soft self mapping which satisfies the following condition:

$$\begin{aligned} \mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b)) &\leq \alpha \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{u}_a), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\} \\ &\quad + \beta \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b)\} \\ &\quad + \gamma \max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\}, \end{aligned}$$

for all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$ with $\hat{u}_a \neq \hat{v}_b$, and for some $\alpha, \beta, \gamma > \tilde{0}$ with $3\alpha + \beta + 3\gamma < \bar{1}$ then, f_φ has a unique soft fixed point $\hat{w}_c \in SP(\tilde{X})$ and moreover for each soft point $\hat{u}_a$ we have $\lim_{n \rightarrow \infty} f_\varphi^n \hat{u}_a = \hat{w}_c$.

Proof: It is enough, if we consider $\psi(\bar{t}) = \bar{t}$ in Theorem 3.3.

Corollary 4.3.6: Let $(\tilde{X}, \mathcal{S}, E)$ be a complete soft S-metric space and let f_φ be soft self mapping on $(\tilde{X}, \mathcal{S}, E)$ which satisfies the following condition:

$$\begin{aligned} &\int_0^{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), f_\varphi(\hat{v}_b))} \xi(t) dt \\ &\leq \alpha \int_0^{\max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{u}_a), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\}} \xi(t) dt + \beta \int_0^{\max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{v}_b)\}} \xi(t) dt \\ &\quad + \gamma \int_0^{\max\{\mathcal{S}(f_\varphi(\hat{u}_a), f_\varphi(\hat{u}_a), \hat{v}_b), \mathcal{S}(f_\varphi(\hat{v}_b), f_\varphi(\hat{v}_b), \hat{u}_a)\}} \xi(t) dt. \end{aligned} \quad (3.11)$$

For all $\hat{u}_a, \hat{v}_b \in SP(\tilde{X})$ with $\hat{u}_a \neq \hat{v}_b$, and for some $\alpha, \beta, \gamma, \eta, \delta > \tilde{0}$ with $3\alpha + \beta + 3\gamma < \bar{1}$, where $\xi: R^+ \rightarrow R^+$ is a Lebesgue-integrable mapping which is summable on compact subset of R^+ , non-negative and such that for each $\epsilon > 0$, $\int_0^\epsilon \xi(t) dt > 0$ then, f_φ has a unique soft fixed point $\hat{w}_c \in SP(\tilde{X})$ and moreover for each soft point $\hat{u}_a$, $\lim_{n \rightarrow \infty} f_\varphi^n \hat{u}_a = \hat{w}_c$.

Proof: If we take $\psi(\bar{t}) = \int_0^{\bar{t}} \xi(t) dt$ in Theorem 3.3, we get desired result.

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