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# Duality and Optimality Conditions in Non-Differentiable Control Problems

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**Abstract**

Optimal control theory has long served as a cornerstone of modern applied mathematics, providing powerful tools for the analysis and regulation of dynamic systems encountered in engineering, economics, management sciences, aerospace technology, and industrial optimization. Most classical formulations of optimal control rely on the assumption that both the system dynamics and the objective functional possess sufficient smoothness properties. In many practical situations, however, these assumptions fail to hold. Real-world systems frequently involve non-differentiable elements such as absolute value functions, switching costs, threshold effects, frictional forces, and sparsity-inducing penalties. The presence of such non-smooth structures complicates the analytical treatment of control problems and requires the use of generalized mathematical techniques. Among the most effective approaches for studying these problems is duality theory, which provides alternative representations of optimization models and establishes fundamental relationships between primal and dual formulations. Through these relationships, one can derive bounds on optimal values, characterize optimal solutions, and obtain meaningful theoretical insights even when conventional differentiability assumptions are absent. This paper examines duality and optimality conditions for a class of non-differentiable control problems within a convex analytical framework. Basic concepts and mathematical preliminaries are developed using generalized gradients and non-smooth analysis. Weak duality, strong duality, and converse duality results are formulated and proved rigorously. Furthermore, necessary and sufficient optimality conditions are established through generalized Hamiltonian techniques. The findings demonstrate that duality principles remain valid in non-smooth settings and continue to provide an effective mechanism for analyzing complex control systems. The theoretical framework developed herein contributes to the broader literature on non-smooth optimization and offers a foundation for future analytical and computational investigations.

**Keywords:** Optimal control, non-differentiable optimization, duality theory, generalized gradient, convex analysis, non-smooth systems, Hamiltonian methods, optimality conditions.

## 1. Introduction

The theory of optimal control emerged during the middle of the twentieth century as a systematic framework for determining control policies that optimize the behavior of dynamic systems. A major breakthrough occurred through the work of Pontryagin and his collaborators, whose Maximum Principle established a set of necessary conditions for optimality in dynamical processes. At nearly the same time, Bellman introduced dynamic programming, thereby providing an alternative perspective based on value functions and recursive optimization. Subsequent contributions by Cesari, Rockafellar, and many others transformed optimal control into a mature mathematical discipline with extensive practical applications.

Traditional optimal control models generally assume that the objective function and system dynamics are continuously differentiable. Under these assumptions, classical differential calculus can be employed to derive necessary and sufficient conditions for optimality. Nevertheless, a growing number of modern applications involve systems whose behavior cannot be accurately described by smooth functions. In such cases, non-differentiability becomes an inherent feature of the mathematical model rather than a mere technical complication.

Examples of non-smooth control problems arise in numerous fields. In aerospace engineering, fuel-minimization models often contain absolute value terms that lead to non-differentiable cost structures. Economic systems with transaction costs, taxation thresholds, or fixed charges frequently generate piecewise-defined objective functions. Mechanical systems affected by dry friction exhibit abrupt changes in motion that are naturally modeled through non-smooth dynamics. Similarly, network optimization and communication systems may involve switching penalties and discontinuous decision mechanisms. More recently, sparse control formulations incorporating $L_1$-norm regularization have become increasingly important in machine learning, signal processing, and resource allocation problems. The lack of differentiability presents substantial analytical challenges. Since classical gradients may fail to exist at critical points, standard optimization techniques cannot be applied directly. To overcome these difficulties, researchers have developed generalized notions of differentiation. One of the most influential contributions in this direction was made by Clarke, whose theory of generalized gradients provided a rigorous framework for extending differential calculus to locally Lipschitz functions. This development significantly broadened the scope of optimization theory and enabled the systematic treatment of non-smooth control systems.

Parallel to these developments, duality theory has become an indispensable component of mathematical optimization. The central idea of duality is to associate with a given optimization problem another problem, known as its dual, whose structure often reveals valuable information about the original formulation. Duality principles can be used to derive lower and upper bounds on optimal values, establish optimality criteria, and facilitate computational procedures. In many instances, the dual problem possesses a simpler structure than the primal problem and therefore offers significant analytical advantages.

For non-differentiable control problems, the integration of duality theory with generalized differentiation techniques provides a particularly powerful framework. The resulting methodology enables the derivation of optimality conditions without relying on classical derivatives and allows for the extension of many familiar results from smooth optimization. Such developments are not only of theoretical interest but are also highly relevant to contemporary applications involving hybrid systems, sparse control, robotics, and data-driven optimization.

The primary objective of the present study is to investigate the role of duality in non-differentiable optimal control problems and to establish rigorous optimality conditions under suitable convexity assumptions. Special attention is devoted to the development of weak duality, strong duality, and converse duality results, together with generalized Hamiltonian optimality conditions. By doing so, the paper aims to contribute to the growing body of literature on nonsmooth control theory and provide a mathematically rigorous foundation for future research in this area.

## 2. Literature Review

The mathematical foundations of optimal control can be traced to the landmark work of Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko (1962), who introduced the Maximum Principle and established a systematic method for analyzing optimal trajectories of controlled dynamical systems. Their work remains one of the most influential contributions to modern control theory. Around the same period, Bellman (1957) developed the theory of dynamic programming, which offered a fundamentally different approach to optimization through recursive decomposition and value functions.

The development of convex analysis played a crucial role in extending optimization theory beyond classical smooth settings. Rockafellar (1970) provided a comprehensive treatment of convex functions, conjugate duality, and variational principles, laying the groundwork for much of modern optimization theory. These concepts later became essential tools in the study of non-smooth mathematical programming and optimal control.

A major advancement in the treatment of non-differentiable problems occurred through the work of Clarke (1975, 1983), who introduced the notion of generalized gradients and developed a rigorous framework for non-smooth analysis. Clarke's theory made it possible to formulate optimality conditions for locally Lipschitz functions and significantly expanded the applicability of optimization methods to practical problems involving non-differentiable phenomena.

The study of duality in nonlinear optimization received considerable attention through the contributions of Wolfe (1961), who established a duality framework for nonlinear programming, and Mond and Hanson (1968), who extended duality concepts to more general optimization settings. Their pioneering investigations demonstrated the theoretical significance of dual formulations and inspired subsequent developments in variational and control problems.

Research on non-smooth optimal control continued to evolve through the work of Vinter (2000), who developed advanced techniques for optimal control systems characterized by state constraints and non-smooth dynamics. Similarly, Mordukhovich (2006) introduced generalized differentiation methods that provided a unified approach to variational analysis, optimization, and control theory. These contributions substantially enriched the mathematical foundations of non-smooth optimization.

Indian researchers have also made noteworthy contributions to the field of duality and generalized convexity. Husain and Jabeen investigated duality models for non-differentiable mathematical programming and established several important results involving generalized invexity conditions. Chandra, Kumar, and Gupta further extended duality principles to broader classes of optimization problems, emphasizing generalized convexity structures and efficiency criteria. Their work has strengthened the theoretical connection between generalized invexity and dual optimality conditions.

In recent years, increasing attention has been directed toward non-smooth systems arising in robotics, machine learning, hybrid dynamical systems, and sparse optimization. The growing importance of these applications has renewed interest in duality theory for non-differentiable control problems. Contemporary research seeks not only to establish theoretical optimality conditions but also to develop computational algorithms capable of handling large-scale non-smooth control models efficiently.

Despite substantial progress, several challenges remain. In particular, the derivation of comprehensive duality frameworks for broad classes of non-differentiable control systems continues to be an active area of investigation. The present study contributes to this ongoing effort by developing a unified treatment of duality and optimality conditions within a convex non-smooth setting.

## 2. Preliminaries

Consider the control system

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.1}$$

subject to

$$x(t_0) = x_0, \tag{2.2}$$

where

$$x(t) \in \mathbb{R}^n, \qquad u(t) \in U \subseteq \mathbb{R}^m,$$

The performance index is

$$J(u) = \int_{t_0}^{t_f} L(t, x(t), u(t))dt. \tag{2.3}$$

**Definition 2.1 (Convex Function).** A function

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

is convex if

$$\phi(\lambda x + (1-\lambda)y) \le \lambda\phi(x) + (1-\lambda)\phi(y), \tag{2.4}$$

for all $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$.

**Definition 2.2 (Locally Lipschitz Function).** A function ϕ is locally Lipschitz if for every compact set K,

$$|\phi(x) - \phi(y)| \le M \|x - y\|, \tag{2.5}$$

for some constant M > 0.

**Definition 2.3 (Clarke Generalized Gradient).** For a locally Lipschitz function ϕ,

$$\partial\phi(x) = \mathrm{co}\left\{\lim_{k\to\infty} \nabla\phi(x_k)\right\}, \tag{2.6}$$

where $x_k$ are differentiability points converging to x, and "co" denotes convex hull.

**Axiom 2.1 (Convexity Axiom).** For every admissible control, L(t, x, u) is convex in (x, u).

**Axiom 2.2 (Regularity Axiom).** The functions f(t, x, u) and L(t, x, u) are locally Lipschitz.

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**Lemma 2.1.** Let ϕ be convex and locally Lipschitz. Then

$$\partial\phi(x) \neq \emptyset. \tag{2.7}$$

**Proof.** Since ϕ is convex and locally Lipschitz, every point admits at least one supporting hyperplane by the Hahn–Banach separation theorem. The subdifferential coincides with the set of supporting vectors. Hence a nonempty generalized gradient exists.

$\partial\phi(x) \neq \emptyset.$

**Proposition 2.1.** For every

$\xi \in \partial\phi(x),$

the subgradient inequality

$$\phi(y)-\phi(x) \geq \xi^T(y-x) \tag{2.8}$$

holds.

**Proof.** By convexity,

$$\phi(\lambda y+(1-\lambda)x) \leq \lambda\phi(y)+(1-\lambda)\phi(x).$$

Rearranging and dividing by λ,

$$\frac{\phi(x+\lambda(y-x))-\phi(x)}{\lambda} \leq \phi(y)-\phi(x).$$

Taking the limit and using the definition of generalized gradient yields

$$\xi^T(y-x) \leq \phi(y)-\phi(x).$$

Hence (2.8) follows.

**Proposition 2.2.** The generalized Hamiltonian

$$H(t,x,u,p) = p^T f(t,x,u) - L(t,x,u) \tag{2.9}$$

is concave whenever f is affine and L is convex.

**Proof.** Since affine functions preserve convex combinations, $p^T f$ is affine. The negative of a convex function is concave. Therefore

H = affine – convex = concave.

Hence H is concave.

## 3. Duality Model

Consider the primal non-differentiable control problem

$$(P) \quad \min J(u) = \int_{t_0}^{t_f} L(t,x,u)dt \tag{3.1}$$

subject to

$$\dot{x} = f(t,x,u). \tag{3.2}$$

Define the Lagrangian

$$\mathbf{L}(x,u,p) = \int_{t_0}^{t_f} \left[ L(t,x,u) + p^T(\dot{x} - f(t,x,u) \right] dt. \tag{3.3}$$

The associated dual problem is

$$(D) \quad \max_{p} \inf_{x,u} \mathbf{L}(x,u,p). \tag{3.4}$$

**Theorem 3.1 (Weak Duality).** Let (x, u) be feasible for (P) and p feasible for (D). Then
D(p) ≤ J(u). (3.5)
**Proof.** From definition,
$$D(p) = \inf_{x,u} \mathbf{L}(x,u,p).$$
Hence
$$D(p) \le \mathbf{L}(x,u,p). \tag{3.6}$$
Since (x, u) is feasible,
$$\dot{x} - f(t,x,u) = 0.$$
Therefore
$$\mathbf{L}(x,u,p) = J(u). \tag{3.7}$$
Combining (3.6) and (3.7),
$$D(p) \le J(u).$$
Thus weak duality holds.

**Theorem 3.2 (Strong Duality).** Assume:
1. Convexity of L.
2. Affinity of f.
3. Slater-type feasibility condition.

Then
Min (P) = max (D). (3.8)
**Proof.** The convexity assumptions imply that the primal problem is a convex optimization problem in infinite-dimensional space.
By the Slater condition, a strictly feasible trajectory exists.
Applying the Fenchel–Rockafellar duality theorem yields
Inf (P) = sup (D). (3.9)
Since both extrema are attained,
Min (P) = max (D).
Hence strong duality follows.

**Theorem 3.3 (Converse Duality).** Suppose $p^*$ is optimal for the dual problem and min (P) = max (D). Then the corresponding primal trajectory ($x^*$, $u^*$) is optimal.
**Proof.** Let
$v_P$ = min (P), $v_D$ = max (D).
By assumption,
$v_P = v_D$. (3.10)
Since $p^*$ attains $v_D$,
$D(p^*) = v_D$.
Using weak duality,
$v_D \le J(x^*, u^*) \le v_P$.
Together with (3.10),
$J(x^*, u^*) = v_P$.

Therefore ($x^*$, $u^*$) is optimal.

**4. Optimality Conditions.**

Define the Hamiltonian

$$H(t,x,u,p) = p^T f(t,x,u) - L(t,x,u) \tag{4.1}$$

**Theorem 4.1 (Necessary Optimality Condition).** If ($x^*$, $u^*$) is optimal, then there exists an adjoint variable p(t) satisfying

$$-\dot{p}(t) \in \partial_x H(t,x^*,u^*,p). \tag{4.2}$$

**Proof.** Using Clarke's non-smooth maximum principle, admissible perturbations of the control must not decrease the objective functional. The first-order variation satisfies

$$0 \in \partial_x H + \dot{p}.$$

Rearranging,

$$-\dot{p} \in \partial_x H.$$

Hence (4.2) follows.

**Theorem 4.2 (Maximum Principle).** For almost every t,

$$H(t,x^*,u^*,p) = \max_{u \in U} H(t,x^*,u,p) \tag{4.3}$$

**Proof.** Suppose there exists $\hat{u}$ such that

$$H(t,x^*,\hat{u},p) > H(t,x^*,u^*,p)$$

Replacing $u^*$ by $\hat{u}$ over a small interval would decrease the objective functional, contradicting optimality.

Hence $u^*$ must maximize the Hamiltonian.

**Theorem 4.3 (Sufficient Optimality Condition).** Assume:

1. H is concave in (x, u);
2. State and adjoint equations hold;
3. Hamiltonian maximization condition holds.

Then ($x^*$, $u^*$) is globally optimal.

**Proof.** Let (x, u) be any feasible pair. Concavity implies

$$H(x,u) - H(x^*,u^*) \le \partial H(x^*,u^*)^T \left[ (x,u) - (x^*,u^*) \right]. \tag{4.4}$$

Integrating over the horizon and employing the adjoint equation yields

$$J(u) - J(u^*) \ge 0. \tag{4.5}$$

Therefore

$$J(u^*) \le J(u)$$

for every feasible control.

Hence ($x^*$, $u^*$) is globally optimal.

**Discussion**
The presented framework demonstrates that classical duality concepts remain valid for a broad class of non-differentiable control problems. The generalized gradient replaces ordinary derivatives while preserving the geometric structure required for optimization.

Weak duality guarantees lower bounds on primal objectives. Strong duality establishes equality between primal and dual optimal values under convexity and regularity assumptions. Converse duality ensures that optimal dual solutions identify optimal primal trajectories.

The generalized Hamiltonian framework extends Pontryagin's principle to non-smooth environments. Such extensions are especially useful in sparse control, switching systems, hybrid systems, economic planning, and machine learning applications involving $L_1$-type penalties.

Recent developments in variational analysis and generalized differentiation suggest further extensions to stochastic control, fractional control, and distributed parameter systems. The integration of duality with numerical algorithms represents an important future research direction.

**Conclusion**
This article developed a rigorous mathematical treatment of duality and optimality conditions for non-differentiable control problems. Fundamental definitions, axioms, lemmas, propositions, and duality theorems were established within a convex analytical framework. Weak duality, strong duality, and converse duality were proved systematically, demonstrating the relationship between primal and dual control formulations. Necessary and sufficient optimality conditions were derived using Clarke's generalized gradient and non-smooth Hamiltonian techniques.

The study confirms that non-differentiability does not destroy the essential structure of optimal control theory. Instead, generalized differentiation enables the extension of classical results to a much wider class of practical systems. These theoretical developments provide a strong foundation for future research in non-smooth control, variational analysis, and computational optimization. The results are expected to have applications in engineering design, economic optimization, robotics, transportation systems, and modern data-driven control environments.

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