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Study of Multipoint Distributions of Growth and Interacting Particle Models: Periodic TASEP and Its Connection to KPZ Universality

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Abstract

The Kardar-Parisi-Zhang (KPZ) universality class governs the large-scale behavior of a vast array of stochastic growth processes in 1+1 dimensions. While single-point height distributions now well characterized, the full spatiotemporal structure—captured by multipoint joint distributions—has only recently become accessible through the periodic very asymmetric simple exclusion process (TASEP). This review provides a comprehensive mathematical treatment of multipoint distributions for periodic TASEP in the critical relaxation time scale $t=O(L^{3/2})$, where finite-size effects compete with KPZ dynamics. The exact present Fredholm determinant formula of Baik and Liu (2019) for the joint height distribution at arbitrary space-time points, and analyze its asymptotic limits. The resulting universal limiting distributions depend on the relaxation parameter $\tau=t\ L^{-3/2}$, interpolating between Tracy-Widom GUE ($\tau\to0$) and Gaussian ($\tau\to\infty$) behavior. Recent extensions to general initial conditions, discrete-time models, and half-space geometries examined, highlighting the role of probabilistic Brownian motion representations. Establish connections to integrable differential equations, including the KP equation, and discuss numerical methods for computing these distributions. This framework provides an integrated view of the two-dimensional KPZ height field, revealing the universal structure of limiting processes with periodic boundary effects.

Keywords: Integrable Probability; Interacting Particle Systems; Universal Height Field; Tracy-Widom Distribution; Bethe Ansatz.

1. Introduction

Random growth models, which describe the evolution of random interfaces in space and time, constitute one of the most fascinating topics in condensed matter physics and probability theory. Since the pioneering work of Kardar, Parisi, and Zhang in 1986 [1], researchers have recognized that a broad class of these models share the same universal scaling behavior, independent of their microscopic details. This behavior is known as the Kardar-Parisi-Zhang (KPZ) universality class, characterized by the 1:2:3 scaling exponents: heights fluctuate on the order of $t^{1/3}$, are correlated spatially over distances of $t^{2/3}$, and temporally over scales of t [2].

Among the most important models in this class is the Totally Asymmetric Simple Exclusion Process (TASEP), which describes particles moving unidirectional on a one-dimensional lattice with a hard-core exclusion rule [3, 4]. The elegance of TASEP lies in its exact solvability using advanced mathematical

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techniques such as the Bethe ansatz and RSK correspondences, enabling the derivation of exact probability distributions quantities. However, understanding the full two-dimensional height field—that is, the joint distributions of growth at different space-time points—remained a significant challenge for many years [5, 6].

The periodic TASEP (PTASEP) has emerged as a powerful tool to address this challenge. In this model, particles placed on a ring of periodic size L, a physically natural setup that simplifies algebraic calculations compared to the infinite domain. The critical regime is the relaxation time scale, where t=O ($L^{3/2}$). On this scale, the spatial correlation length $t^{2/3}$ becomes comparable to the ring size L, leading to "critical interactions" among particles and a strong effect of periodic boundaries on the limiting distributions [7, 8].

Despite the relatively simple nature of periodic TASEP, the one-point distributions only derived recently, through independent works in physics [9] and mathematics [10]. These works demonstrated that the limiting distributions follow KPZ scaling but differ from those of the infinite domain, reflecting the effect of finite geometry [11].

This research aims to provide an analytical review of the methods and results concerning the computation of multipoint distributions for periodic TASEP and their connection to KPZ universality. The focus will be on the mathematical foundations, main results, recent developments, and future prospects.
While single-point height distributions in KPZ models extensively characterized since Johansson's seminal work (2000), the full probabilistic structure of the height field—specifically, joint distributions at multiple space-time points—remained largely unexplored until recently. Multipoint distributions are essential for understanding:

- Spatiotemporal correlations: The decay of correlations across space and time
- Covariance structure: The full two-point function and beyond
- Scaling limits: The nature of the limiting universal process
- Finite-size effects: How boundaries and geometry modify universal behavior"

Despite significant progress on infinite-domain TASEP, three fundamental questions remained unresolved prior to 2019:
1. What are the exact multipoint distributions for TASEP with periodic boundary conditions?
2. How do these distributions behave in the critical relaxation regime?
3. What universal process emerges as the scaling limit, and how does it interpolate between known limits?

This review addresses these questions by systematically presenting the mathematical framework and key results.

2. Literature Review: Evolution of Methods and Models

The history of studying interacting particle models in the KPZ class spans more than three decades, with cumulative progress from exact solutions of simple models to the construction of universal limiting processes.

2.1. Foundations: TASEP on the Infinite Domain

The infinite TASEP model formed the foundation upon which most subsequent developments built. The exact solution for the transition probability in this model is due to Schütz [16], who employed the coordinate Bethe ansatz to solve the forward Kolmogorov equation, transforming it into a free evolution equation with specific boundary conditions.

Schütz's solution of the transition probability employs the coordinate Bethe ansatz, which assumes the wave function takes the form:

$\Psi_{\mathbf{x}}(\mathrm{y}, t) = \sum_{\sigma \in S_N} A_\sigma\, e^{i} \sum_j {}^{k\sigma(j)yj}\, e^{-E(\mathbf{k})t}$ (1)

Where N is the number of particles, x and y are initial and final positions, S_N is the symmetric group, and the energy $E(\mathrm{k}) = \sum_j (1-e^{-ikj})$. The Bethe equations impose periodicity conditions leading to the quantization of momenta:

$E^{ikj\,L} = \prod_{\ell=j} (e^{ik\ell} + e^{ikj} - 2e^{ikj}\, e^{ik\ell})/(e^{ikj} + e^{ik\ell} - 2e^{ikj}e^{ik\ell})$ (2)

In the early 2000s, Johansson [12] derived the celebrated Fredholm determinant formula for the one-point distribution of TASEP under step initial conditions, opening the door to scaling analysis and the extraction of the Tracy-Widom GUE distribution. This following by generalizations to multipoint distributions in the spatial direction through works of Sasamoto [17], Borodin, and collaborators [18]. However, generalizations to points in the temporal direction remained elusive for a considerable period.

2.2. The Turning Point: Periodic TASEP

The emergence of periodic TASEP in physics studies represented a crucial turning point. In 2016-2017, Prolhac [19] published physics results on the one-point distribution of the periodic model in the relaxation time scale, followed by independent mathematical works by Baik and Liu [10] and Liu [20] providing rigorous proofs of the same results for step, flat, and stationary initial conditions.

The relaxation time scale $t=O(L^{3/2})$ represents the critical regime where KPZ dynamics competes with finite-size effects. On this scale, all particles on the ring become critically correlated, leading to limiting distributions that depend on the relaxation parameter $\tau = t\ L^{-3/2}$ and differ from the infinite-domain distributions [13].

2.3. The Major Achievement: Multipoint Distributions

In 2019, Baik and Liu [13] achieved a qualitative breakthrough with their publication in the Journal of the American Mathematical Society. They presented exact formulas for multipoint distributions (multi-time and multi-space) of periodic TASEP under the periodic step initial condition. Their systematic approach involved:

*Step 1: Transition Probability Computation***.** Starting from Schütz's formula, they derived the periodic analogue by imposing periodic boundary conditions $\eta_j + L = \eta_j$. This yielded:

$P(\eta^t = \eta' | \eta^0 = \eta) = 1/L^N \sum_k \det[e^{ikj(x\ell - y\ell)}]\ \det[e^{-ikjx\ell}]e^{-tE(k)}$ (3)

Step 2: Summation over Final Positions. The joint distribution $P(x^{(1)}(t1), \ldots, x^{(m)}(tm))$ was expressed as a multiple sum over intermediate positions, exploiting the Markov property.

Step 3: Determinant Simplification. For the periodic step initial condition (particles occupying positions 1, 2,…,N), the multiple sum reduced to a single Fredholm determinant:

$P(\cap_{i-1}^{m} \{X_i \le s_i\}) = \det(I - k_{s,t}^{(m)})$ (4)

Where: $k_{s,t}^{(m)}$ is an integral operator on $L^2(R)$ with kernel:

$K(m)(u,v)= \frac{1}{2\pi i}\int_C dz e^{z(u-v)} \prod_{j=1}^{m} 1/(1+e^{-(sj-ztj)})$ (5)

*Step 4: Asymptotic Analysis***.** The method of steepest descent was applied in the limit t, L→∞ with $\tau= t\, L^{-3/2}$ fixed. This involved deforming contours through saddle points and analyzing the leading-order behavior of the Fredholm determinant.

2.4. Subsequent Developments and Generalizations

The work of Baik and Liu following by a series of significant developments:

General Initial Conditions (2021): Baik and Liu [21] generalized their results to general initial conditions, including flat and step-flat conditions, proving their convergence to limiting distributions depending on the relaxation parameter.

Discrete-Time Model (2022): Liao [9] studied the discrete-time TASEP with parallel update on a periodic domain, obtaining analogous formulas for multipoint distributions and proving their limits in the relaxation time scale for step and flat conditions.

Half-Space Domain (2025): Zhang [15] studied TASEP in the half-space with general boundary conditions, providing Pfaffian determinant formulas for multipoint distributions and applying the 1:2:3 scaling to extract KPZ fixed-point distributions in the half-space.

Probabilistic Representation and KPZ Fixed Point (2025): Liao and Liu [14] introduced an innovative probabilistic representation of the kernel for multipoint distributions of the KPZ fixed point using expectations of colliding Brownian motions, paving the way for a more comprehensive understanding of the global field on the periodic domain.

3. Analytical Framework: Mathematical Foundations and Key Results

A. Physical Interpretation of the Relaxation Parameter

The relaxation parameter τ admits a physical interpretation. The diffusion constant for TASEP is $D=|1-2\rho|$ (where $\rho=N/L$ is the density), and the relaxation time for the ring is $\tau_{relax}=L^2/D$. The condition $t=O(L^{3/2})$ corresponds to $t/\tau_{relax} =O(L^{-1/2}) \to 0$, meaning the observation time is still much smaller than the relaxation time. However, the KPZ correlation length $\xi(t)\sim t^{2/3}$ becomes comparable to L, creating a critical competition between KPZ dynamics and finite-size effects.

B. Expanded Theorem Statement

Theorem (Baik & Liu, 2019, Theorem 1.1): Consider PTASEP on a ring of size L with N particles under the periodic step initial condition. For any $m\in N$, times $0<t1<\cdots<tm$ and integers $a_1,\ldots, am$, the joint distribution of the height function satisfies:

$P\left(\cap_{i-1}^{m} \{H(ai, ti)\le hi\}\right)=\det(I- K_{(ai,ti,hi)}^{(m)})$ (6)

Where $K^{(m)}$ is a trace-class operator on L^2® with kernel defined by a contour integral. The coefficient given explicitly through:

$(K^{(m)} f)(x)= \sum_{k=1}^{m}\int_R dy K_k(x,y)f(y)$ (7)

With

$K_k(x,y)=\frac{1}{2\pi i}\int_{Ck} d\lambda e^{\lambda(x-y)} \prod_{j=1}^{m} 1/(1+e^{-(h_j - a_j \log\lambda - t_j \phi(\lambda))})$ (8)

Where $\phi(z) = \frac{z}{1-z}$ and the contours C_k are chosen to ensure convergence.

C. Detailed Asymptotic Analysis

Steepest Descent Implementation:

The method of steepest descent applied to the contour integrals in the Fredholm determinant. The key observation is that in the limit $t\sim L^{3/2}$, the exponent $t\phi(z)$ must be balanced with $a_i \log z$, leading to the scaling of spatial variables:

$a_i = O(t^{2/3})$, $h_i = O(t^{1/3})$.

Under this scaling, the saddle points z_j are solutions to:

$\phi'(z_j) = a_i/t = O(t^{-1/3})$

The leading contribution yields the limiting kernel:

$K_\tau^{(m)}(u,v)=\frac{1}{2\pi i}\int_{C\infty} dz\, e^{z(u-v)} \prod_{j=1}^{m} (1/(1+e^{-(xj-yj\, z2/3-\tau z3/2)}))$ (9)

This kernel defines the periodic Airy process, which is the universal limiting process for PTASEP in the relaxation regime.

D. Interpolation Behavior

✓ Short-Time Limit ($\tau\rightarrow 0$):

As $\tau\rightarrow 0$, the kernel reduces to the standard Airy kernel:

$K_\tau^{(m)}(u,v)\rightarrow K_{Airy}(u,v)=\int_{\infty}^{0} d\lambda\, Ai(u+\lambda)Ai(v+\lambda)$ (10)

Consequently, the limiting distribution becomes:

$F(x; 0, \gamma) = F_{TW\text{-}GUE}(x) = \det(I-K_{Airy})_{(x,\infty)}$ (11)

This rigorously proves that finite-size effects vanish at short times, recovering the infinite-domain result

✓ Long-Time Limit ($\tau\rightarrow\infty\tau\rightarrow\infty$):

For $\tau\rightarrow\infty$, the kernel becomes:

$K_\tau^{(m)}(u,v)\rightarrow \frac{1}{2\pi i}\int_{C\infty} dz\, e^{z(u-v)} \prod_{j=1}^{m} (1/(1+e^{-(xj-\tau z3/2)})$ (12)

Analysis of this kernel yields convergence to the Gaussian distribution:

$F(x;\infty,\gamma)=\Phi((x-\mu(\tau,\gamma))/\sigma(\tau,\gamma))$ (13)

4. Connections to Integrable Systems

The limiting distributions obtained for periodic TASEP exhibit deep connections to integrable differential equations. Baik et al. (2020) demonstrated that the one-point limiting distribution $F(x;\tau,\gamma)$ satisfies the KP equation:

$\frac{\partial}{x\,\partial}(\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} - (\partial^3 u/\partial x^3)) = 0$ (14)

Where $u(x,t)=-2\,\partial_x^2 \log F(x;t,\gamma)$. This connects the probabilistic problem to the theory of integrable systems.

Implications:

1. Solation Solutions: The KP equation admits solution solutions that may correspond to special cases of the height field.

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2. Integrability: This provides a rigorous mathematical framework for analyzing the limiting distributions.
3. Generalization: This suggests that other KPZ-class models governed by integrable equations.

Open Problem: Whether the full multipoint distribution satisfies, a hierarchy of integrable equations remains an active area of investigation.

5. Numerical Methods for Fredholm Determinants

The numerical evaluation of Fredholm determinants presents both opportunities and challenges. Several approaches developed:

- ✓ Nyström Method: The Fredholm determinant approximated by discretizing the integral operator. For a kernel $K(x, y)$ on $[a, b]$, the Nyström approximation is:

$$\det(I-K) \approx \det(\delta_{ij} - w_j K(x_i, x_j)) \quad (15)$$

Where $\{x_i\}$ are quadrature points and $\{w_i\}$ are quadrature weights.

- ✓ Born Series Expansion: For small τ or large x, the determinant computed via expansion:

$$\det(I-K) = \exp\left(-\sum_{n=1}^{\infty} 1/n\,(\mathrm{Tr}(K^n))\right) \quad (16)$$

- ✓ Monte Carlo Methods: For high-dimensional multipoint distributions, Monte Carlo integration of the underlying particle system provides a verification tool.
- ✓ Convergence and Error Analysis: The convergence rate of Nyström methods for integral operators with analytic kernels is exponential. However, the kernels in periodic TASEP are only piecewise analytic due to branch cuts, necessitating careful treatment.

6. Key Mathematical Mechanisms

The exact formulas for multipoint distributions rely on a set of advanced mathematical tools:

1. Bethe Ansatz Method**:** Used to solve the forward Kolmogorov equation of the system, transforming it into a free evolution equation with specific boundary conditions, yielding an exact formula for the particle transition probability [13, 16].
2. Fredholm Determinants: The primary mechanism for reducing multiple-sum formulas to operator-theoretic formulas amenable to asymptotic analysis [12; 13].
3. Method of Steepest Descent: Used to extract asymptotic limits of integrals and Fredholm determinants in the 1:2:3 scaling [13; 22].
4. Brownian Motion Representation: Recently employed to represent the kernel carrying initial condition information, enabling generalization to general initial conditions [14].

7. Open Problems

- ✓ The $\tau \to 0$ Limit for Multipoint Distributions: While the one-point limit established, the rigorous connection between multipoint periodic distributions and their infinite-domain counterparts remains incomplete.
- ✓ Full Two-Point Function: The computation of the full covariance structure C(s,t) =E[H(x,s) H(y,t)]− E[H(x,s)] E[H(y,t)] has not been fully achieved for general densities.
- ✓ Discrete-Time Extensions: While Liao (2022) established results for discrete-time parallel update, the generalization to other update schemes remains open.
- ✓ Long-Range Interactions: Models with longer-range interactions (e.g., multi-particle jumps) have not been investigated in the periodic setting.

- ✓ Higher Dimensions: Generalization to higher-dimensional KPZ models on periodic domains is entirely unexplored.
- ✓ Connection to Random Matrix Theory: The explicit link between periodic TASEP distributions and random matrix ensembles beyond the Tracy-Widom family remains elusive

8. Physical Applications and Implications

- ✓ Biological Transport: TASEP models applied to protein synthesis and intracellular transport. The periodic geometry corresponds to circular DNA or cellular compartments.
- ✓ Traffic Flow: The exclusion process models vehicular traffic on circular roads, with the relaxation time scale corresponding to the transition between free flow and congestion.
- ✓ Surface Growth: The height function corresponds to interface roughness, with periodic boundary conditions modeling deposition on cylindrical substrates.
- ✓ Verification of KPZ Scaling: The periodic TASEP provides a controlled setting for testing universal scaling predictions, with the relaxation parameter controlling the crossover between universality classes.

9. Conclusion

Recent research has demonstrated that periodic TASEP provides an ideal framework for studying the universal structure of height fields in the KPZ class with periodic boundary effects. The main findings summarized as follows:

1. *Exact Formulas*: Exact formulas for multipoint distributions in space-time for PTASEP derived in the form of multiple integrals involving Fredholm determinants, valid for any number of space-time points under general initial conditions [13, 14].
2. *Scaling Limits:* In the relaxation time scale $t=O(L^{3/2})$, these distributions converge to limiting distributions depending on the relaxation parameter τ, differing from infinite-domain distributions. This represents the periodic analogue of the KPZ fixed point [13; 22].
3. *Interpolating Behavior:* The limiting distributions interpolate between the Tracy-Widom GUE distribution in the short-time limit and the Gaussian distribution in the long-time limit, reflecting the transition from KPZ dynamics to finite-system stationary dynamics [22].
4. *Generalizations*: These results have been extended to multiple initial conditions (step, flat, step-flat, random) [21], to discrete-time models [9], to the half-space domain [15], and with novel probabilistic kernel representations using Brownian motion [14].

10. Recommendations

Based on the reviewed results, the following recommendations for future research proposed:

1. *Explore the $\tau\to 0$ Limit:* Investigate the $\tau\to 0$ limit of multipoint distributions to rigorously connect periodic distributions with those of the infinite domain, a connection not yet fully resolved.
2. *Expand the Model Family:* Generalize the results to broader families of KPZ models such as ASEP with asymmetric jumps, directed polymer models at finite temperature, and growth models with longer-range interactions.
3. *Higher-Order Spacetime Correlations:* Study deeper statistical properties of the periodic height field, such as higher-order (more than two points) spacetime correlations, full correlation functions, and detailed probabilistic structures.

4. *Numerical Applications:* Develop efficient numerical algorithms for computing Fredholm determinants and multipoint distributions numerically, to verify theoretical predictions and compare with direct simulations.
5. *Connections to Differential Equations:* Investigate the relationship between limiting distributions and integro-differential equations such as the KP equation, coupled mKdV systems, and the KdV equation, as suggested by Baik and colleagues [22].

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