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Bicomplex Hilbert Spaces and Their Applications

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Abstract:
Complex inner-product geometry is extended to modules over the bicomplex algebra by bicomplex Hilbert spaces. However, positivity is inherently hyperbolic, and zero divisors influence normalisation, independence, and spectral interpretation. This paper delineates a rigorous componentwise theory. In the hyperbolic cone, a bicomplex inner product is defined as a positive quadratic value that is obtained through idempotent conjugation. The componentwise square root is the norm that is associated. The bicomplex Cauchy–Schwarz and triangle inequalities, the projection theorem for closed submodules, and the Riesz representation theorem for continuous bicomplex-linear functionals are all established through detailed proofs. Under explicit component-dimension hypotheses, orthogonality, Pythagorean identities, orthogonal complements, Gram–Schmidt procedures, Bessel inequalities, Parseval identities, and basis expansions are derived. Division by a null-cone norm is invalid, and free orthonormal bases necessitate matched complex dimensions, as explained by the Gram–Schmidt analysis. Componentwise characterisations of self-adjoint, unitary, positive, compact, and normal operators are provided, and the normal spectral theorem is summarised in an idempotent form. Finite-dimensional modules, projections, L^2 spaces, and bicomplex reproducing-kernel Hilbert spaces are among the examples. The critical evaluation of applications to bicomplex quantum mechanics, Fourier representations, signal processing, frames, and analytic kernels is conducted. The theory provides precise mathematics for paired complex channels, whereas physical probabilities and genuinely coupled models necessitate additional axioms. Unbounded observables, C*-modules, frames with unequal component dimensions, tensor products, and spectral measures on bicomplex domains are all open problems.

Keywords: bicomplex Hilbert space; hyperbolic norm; Cauchy–Schwarz inequality; projection theorem; Riesz representation; orthonormal basis; adjoint operator; quantum mechanics

Mathematical Notation and Symbols

Symbol	Meaning
$H=\mathbf{e}_1H_1 \oplus \mathbf{e}_2H_2$	Bicomplex Hilbert module with complex Hilbert components.
Z^{†3}	Conjugation $\bar{Z}_1\mathbf{e}_1+\bar{Z}_2\mathbf{e}_2$.
$\langle x,y\rangle\mathscr{B}\mathbb{C}$	Bicomplex inner product, conjugate-linear in x and linear in y.

Symbol	Meaning
$\|x\|_{\mathbb{D}}$	Hyperbolic norm $\sqrt{\langle x,x\rangle}$.
$M\perp$	Orthogonal complement of a submodule M.
P_M	Orthogonal projection onto M.
T*	Adjoint of a bounded operator T.
{u_r}	Bicomplex orthonormal system when $\langle u_r,u_s\rangle=\delta rs$.

1. Introduction

Hilbert-space methods combine algebra, geometry, and completeness. In the bicomplex setting the algebra has two complex idempotent components, so inner products and norms must preserve both while respecting zero divisors. Finite- and infinite-dimensional bicomplex Hilbert spaces were developed systematically by Lavoie, Marchildon, and Rochon (2010, 2011), and subsequent work studied bicomplex operators, analytic spaces, and frames (Kumar & Singh, 2015; Elgourari et al., 2021).

The central structural fact is $H=\mathbf{e}_1H_1\oplus\mathbf{e}_2H_2$. Most inequalities are therefore exact pairs of complex inequalities. This does not make every issue trivial. A nonzero pure vector has a norm in the null cone, so its norm cannot be inverted as a bicomplex scalar. A bicomplex orthonormal vector must have unit length in both components, and a free orthonormal basis exists only when the component Hilbert dimensions match. Spectra and quantum probabilities similarly require conventions beyond formal decomposition.

The paper proves the main Hilbert theorems with these qualifications explicit. It then discusses operator classes, reproducing kernels, Fourier and frame representations, signal models, and bicomplex quantum mechanics. Established component theorems are used transparently, and interpretive claims are separated from mathematical identities.

2. Bicomplex Inner Products and Positivity

Definition 2.1. Let $H=\mathbf{e}_1H_1\oplus\mathbf{e}_2H_2$. A bicomplex inner product is

$$\langle x,y\rangle_{\mathscr{B}\mathbb{C}}=\langle x_1,y_1\rangle_1\mathbf{e}_1+\langle x_2,y_2\rangle_2\mathbf{e}_2, \tag{2.1}$$

where each component inner product is conjugate-linear in the first argument and linear in the second. Thus $\langle x,y\rangle=\langle y,x\rangle$^{†3}, $\langle x,\alpha y+\beta z\rangle=\alpha\langle x,y\rangle+\beta\langle x,z\rangle$, and $\langle x,x\rangle\in\mathbb{D}^+$ with equality to zero exactly when $x=0$.

Definition 2.2. The associated hyperbolic norm is

$$\|x\|_{\mathbb{D}}=\sqrt{\langle x,x\rangle}=\|x_1\|_1\mathbf{e}_1+\|x_2\|_2\mathbf{e}_2. \tag{2.2}$$

The square root is taken componentwise. If $x=\mathbf{e}_1x_1\neq0$, then $\|x\|_{\mathbb{D}}=\|x_1\|\mathbf{e}_1$ is nonzero but not invertible. Positivity therefore means membership in $\mathbb{D}^+$, not invertibility.

Definition 2.3. Vectors x and y are orthogonal, written $x\perp y$, when $\langle x,y\rangle=0$. A family {u_r} is orthonormal when $\langle u_r,u_s\rangle=\delta rs\cdot1$.

Proposition 2.4. Orthogonality is componentwise: $x\perp y$ if and only if $x_1\perp y_1$ and $x_2\perp y_2$. For $x\perp y$,

$$\|x+y\|_{\mathbb{D}}^2=\|x\|_{\mathbb{D}}^2+\|y\|_{\mathbb{D}}^2. \tag{2.3}$$

Proof. The inner product vanishes exactly when both coefficients vanish. Expanding $\langle x+y,x+y\rangle$ and using conjugate symmetry eliminates the two cross terms. □

3. Fundamental Inequalities

Theorem 3.1. Bicomplex Cauchy–Schwarz inequality. For all $x,y\in H$,

$$|\langle x,y\rangle_{\mathscr{B}\mathbb{C}}|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|x\|_{\mathbb{D}}\|y\|_{\mathbb{D}}. \qquad (3.1)$$

Proof. Write $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$ and $y=y_1\mathbf{e}_1+y_2\mathbf{e}_2$. The complex Cauchy–Schwarz inequality gives $|\langle x_j,y_j\rangle_j|\leq\|x_j\|_j\|y_j\|_j$ for $j=1,2$. Therefore $|\langle x,y\rangle|_{\mathbb{D}}=|\langle x_1,y_1\rangle|\mathbf{e}_1+|\langle x_2,y_2\rangle|\mathbf{e}_2\preccurlyeq_{\mathbb{D}}\|x_1\|\|y_1\|\mathbf{e}_1+\|x_2\|\|y_2\|\mathbf{e}_2$. The right side is exactly $(\|x_1\|\mathbf{e}_1+\|x_2\|\mathbf{e}_2)(\|y_1\|\mathbf{e}_1+\|y_2\|\mathbf{e}_2)$, because cross-idempotent products vanish. This proof never divides by $\|y\|_{\mathbb{D}}$ and remains valid when y has a null-cone norm. □

Theorem 3.2. Triangle inequality. For all $x,y\in H$,

$$\|x+y\|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|x\|_{\mathbb{D}}+\|y\|_{\mathbb{D}}. \qquad (3.2)$$

Proof. Expand $\|x+y\|_{\mathbb{D}}^2=\|x\|_{\mathbb{D}}^2+2\mathrm{Re}_{\mathbb{D}}\langle x,y\rangle+\|y\|_{\mathbb{D}}^2$, where $\mathrm{Re}_{\mathbb{D}}$ acts componentwise. Theorem 3.1 gives $\mathrm{Re}_{\mathbb{D}}\langle x,y\rangle\preccurlyeq_{\mathbb{D}}|\langle x,y\rangle|_{\mathbb{D}}\preccurlyeq_{\mathbb{D}}\|x\|_{\mathbb{D}}\|y\|_{\mathbb{D}}$. Hence $\|x+y\|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}(\|x\|_{\mathbb{D}}+\|y\|_{\mathbb{D}})^2$. Componentwise square roots are monotone on $\mathbb{R}_+$, proving the result. □

Proposition 3.3. The parallelogram identity holds:

$$\|x+y\|_{\mathbb{D}}^2+\|x-y\|_{\mathbb{D}}^2=2\|x\|_{\mathbb{D}}^2+2\|y\|_{\mathbb{D}}^2. \qquad (3.3)$$

Proof. Expand the two inner products. The cross terms cancel in each complex component. □

Corollary 3.4. The map $\|\cdot\|_{\mathbb{D}}$ induces the product Hilbert topology, and H is complete if and only if H_1 and H_2 are complex Hilbert spaces.

Proof. The norm coefficients are the component norms. Convergence and Cauchy behavior are therefore componentwise, so completeness is equivalent. □

4. Orthonormal Systems and Gram–Schmidt

Theorem 4.1. Bicomplex Gram–Schmidt for free sequences. Suppose $x_1,\ldots,x_m$ is bicomplex-linearly independent, equivalently both component families $x_{1,j},\ldots,x_{m,j}$ are complex-linearly independent. Then the component Gram–Schmidt processes produce $u_1,\ldots,u_m$ with $\langle u_r,u_s\rangle=\delta_{rs}$ and the same generated submodule.

Proof. At step r, define $v_{r,j}=x_{r,j}-\Sigma_{s<r}u_{s,j}\langle u_{s,j},x_{r,j}\rangle$. Component independence ensures $v_{r,j}\neq 0$. Put $u_{r,j}=v_{r,j}/\|v_{r,j}\|$ and $u_r=u_{r,1}\mathbf{e}_1+u_{r,2}\mathbf{e}_2$. Then $\langle u_r,u_s\rangle$ has coefficients δ_{rs}, so it equals $\delta_{rs}\cdot 1$. The usual triangular relations give equality of spans in each component and hence of bicomplex submodules. The combined residual norm has two positive coefficients and is invertible; this is exactly where the free-independence hypothesis is needed. □

Remark 4.2. For a general generating family, one component residual may vanish before the other. Dividing by the hyperbolic norm would then be invalid. The correct procedure is to continue Gram–Schmidt separately in H_1 and H_2, yielding idempotent orthonormal systems that need not pair into a free bicomplex basis.

Theorem 4.3. Bessel inequality. For a finite or countable bicomplex orthonormal system $\{u_r\}$,

$$\Sigma_r\,|\langle u_r,x\rangle|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}\|x\|_{\mathbb{D}}^2. \qquad (4.1)$$

Proof. Each component $\{u_{r,j}\}$ is a complex orthonormal system. Complex Bessel inequalities give $\Sigma_r|\langle u_{r,j},x_j\rangle|^2\leq\|x_j\|^2$. Multiply by $\mathbf{e}_j$ and add. Monotone convergence handles a countable system componentwise. □

Theorem 4.4. Parseval and basis expansion. If $\{u_r\}_{r\in I}$ is a bicomplex orthonormal basis, meaning both component families are complete orthonormal bases with the same index set, then

$$x=\Sigma_{r\in I}u_r\langle u_r,x\rangle,\quad \|x\|_{\mathbb{D}}^2=\Sigma_{r\in I}|\langle u_r,x\rangle|_{\mathbb{D}}^2. \qquad (4.2)$$

Proof. Apply the complex basis expansion and Parseval identity in H_1 and H_2. Both series converge in their component norms; recombination gives convergence in H. The common index requirement reflects equality of Hilbert dimensions for a free bicomplex orthonormal basis. □

Example 4.5. Standard finite space. On $H=\mathscr{B}\mathbb{C}^m$, define $\langle x,y\rangle=\Sigma$_k x_k^{†3}y_k. The standard vectors form a bicomplex orthonormal basis, and Gram–Schmidt is exactly two complex Gram–Schmidt algorithms.

Proof. The jth coefficient is the standard inner product on $\mathbb{C}^m$. Positivity, completeness, and orthonormality follow componentwise. □

5. Projection and Riesz Representation

Definition 5.1. For a submodule $M\subset H$, its orthogonal complement is $M\perp=\{x:\langle m,x\rangle=0$ for every $m\in M\}$.

Proposition 5.2. $M\perp$ is a closed submodule and $(M\perp)\perp$ equals the closure of M.

Proof. Linearity and conjugate symmetry show that $M\perp$ is a submodule. It is the intersection of kernels of continuous functionals $x\mapsto\langle m,x\rangle$ and is therefore closed. The double-complement identity holds in both complex components and recombines. □

Theorem 5.3. Projection theorem. If M is a closed bicomplex submodule of H, then every $x\in H$ has a unique decomposition

$$x=m+n,\quad m\in M,\ n\in M\perp, \tag{5.1}$$

and m=P_Mx is the unique nearest point in the componentwise sense.

Proof. Write $M=M_1\mathbf{e}_1\oplus M_2\mathbf{e}_2$. Closedness in the product topology makes each M_j closed. By the complex projection theorem, $x_j=m_j+n_j$ uniquely with $m_j\in M_j$ and $n_j\in M_j\perp$. Set $m=m_1\mathbf{e}_1+m_2\mathbf{e}_2$ and $n=n_1\mathbf{e}_1+n_2\mathbf{e}_2$. Then $n\in M\perp$ and $x=m+n$. If another decomposition exists, subtracting gives a vector in $M\cap M\perp$, whose norm square is its inner product with itself and must be zero; uniqueness follows. For any $y\in M$, $x-y=n+(m-y)$ with orthogonal terms, so $\|x-y\|\mathbb{D}^2=\|n\|\mathbb{D}^2+\|m-y\|\mathbb{D}^2\succcurlyeq\mathbb{D}\|n\|\mathbb{D}^2$, with equality only at $y=m$. □

Corollary 5.4. P_M is bounded, self-adjoint, and idempotent; $H=M\oplus M\perp$.

Proof. The component projections have norm one unless the component subspace is zero, and are self-adjoint idempotents. Recombine. □

Theorem 5.5. Componentwise Riesz representation. For every continuous bicomplex-linear functional $f:H\to\mathscr{B}\mathbb{C}$, there is a unique $u\in H$ such that

$$f(x)=\langle u,x\rangle\mathscr{B}\mathbb{C} \text{ for all } x,\quad \text{and}\quad \|f\|\mathbb{D}=\|u\|\mathbb{D}. \tag{5.2}$$

Proof. Decompose $f=f_1\mathbf{e}_1+f_2\mathbf{e}_2$. Each f_j is a continuous complex-linear functional on H_j. By the complex Riesz theorem there is a unique u_j with $f_j(x_j)=\langle u_j,x_j\rangle$ and $\|f_j\|=\|u_j\|$. Put $u=u_1\mathbf{e}_1+u_2\mathbf{e}_2$. Equation (2.1) gives $f(x)=\langle u,x\rangle$. If v represents f, then $\langle u-v,x\rangle=0$ for every x; taking $x=u-v$ gives $u=v$. The norm identity is the pair of complex identities. □

6. Adjoint and Spectral Operators

Theorem 6.1. Every bounded bicomplex-linear $T:H\to H$ has a unique adjoint $T=T_1\mathbf{e}_1+T_2\boldsymbol{e}_2$ *satisfying* $\langle Tx,y\rangle=\langle x,Ty\rangle$. Moreover $\|T\|\mathbb{D}=\|T\|\mathbb{D}$ *and* $(ST)=TS$.

Proof. For fixed y, $x\mapsto\langle Tx,y\rangle$ is conjugate-linear in x; equivalently use the component complex adjoints to define T*. The identities hold in each component and hence in H. □

Proposition 6.2. T is self-adjoint, positive, normal, isometric, or unitary if and only if both T_1 and T_2 are. A self-adjoint T has real component spectra, while the full bicomplex spectrum remains cylindrical.
Proof. Every defining identity splits componentwise. Complex self-adjoint spectra lie in $\mathbb{R}$, so the reduced spectrum lies in $\mathbb{D}$; failure of invertibility in either component still generates a cylinder in the full bicomplex spectrum. □
Theorem 6.3. Normal spectral representation. If T is bounded and normal, then for bounded Borel functions g_1 on $\sigma(T_1)$ and g_2 on $\sigma(T_2)$,

$$g(T)=g_1(T_1)\mathbf{e}_1+g_2(T_2)\mathbf{e}_2 \qquad (6.1)$$

defines a componentwise functional calculus, with $\|g(T)\|_{\mathbb{D}}=\|g_1(T_1)\|\mathbf{e}_1+\|g_2(T_2)\|\mathbf{e}_2$.
Proof. Apply the complex Borel functional calculus to the normal components. Algebraic, adjoint, and norm properties recombine. The domain of g is explicitly the pair of component spectra, avoiding ambiguity from the full cylindrical spectrum. □

6.4 Polarization, Compact Spectra, and Uncertainty

Theorem 6.4. Polarization identity. The bicomplex inner product is recovered from its hyperbolic norm by

$$\langle x,y\rangle=\tfrac{1}{4}[\|x+y\|_{\mathbb{D}}^2-\|x-y\|_{\mathbb{D}}^2-\mathbf{i}\|x+\mathbf{i}y\|_{\mathbb{D}}^2+\mathbf{i}\|x-\mathbf{i}y\|_{\mathbb{D}}^2]. \qquad (6.2)$$

Proof. Apply the complex polarization identity for an inner product conjugate-linear in the first argument and linear in the second to H_1 and H_2. The scalar $\mathbf{i}$ has the same complex coordinate in both idempotent components, so the two identities recombine. This also shows that a hyperbolic norm satisfying the component parallelogram law determines the inner product uniquely. □
Corollary 6.5. A bicomplex-linear isometry U with U(0)=0 preserves the inner product. A surjective such map is unitary.
Proof. If $\|Ux\|_{\mathbb{D}}=\|x\|_{\mathbb{D}}$, the four norm terms in the polarization identity are preserved, so $\langle Ux,Uy\rangle=\langle x,y\rangle$. Then U*U=I; surjectivity also gives UU*=I. □
Theorem 6.6. Compact self-adjoint spectral theorem. Let T be compact and self-adjoint. Each component H_j has an orthonormal basis consisting of eigenvectors of T_j together with a basis of ker T_j, and the nonzero component eigenvalues are real, finite-multiplicity, and tend to zero. Accordingly,

$$Tx=\Sigma_r \lambda_{r,1}\langle u_{r,1},x_1\rangle u_{r,1}\mathbf{e}_1+\Sigma_s \lambda_{s,2}\langle u_{s,2},x_2\rangle u_{s,2}\mathbf{e}_2. \qquad (6.3)$$

Proof. This is the complex compact self-adjoint spectral theorem applied to T_1 and T_2. Each component series converges in norm. The two index sets need not have the same cardinality, so the safest expansion is the separated idempotent formula rather than an artificially paired list. □
Definition 6.7. For a normalized bicomplex vector x with $\langle x,x\rangle=1$ and a bounded self-adjoint A, define the expectation $a_x=\langle x,Ax\rangle\in\mathbb{D}$ and uncertainty

$$\Delta_x(A)=\|(A-a_xI)x\|_{\mathbb{D}}. \qquad (6.4)$$

Theorem 6.8. Robertson-type uncertainty inequality. For bounded self-adjoint A and B and normalized x,

$$\Delta_x(A)\Delta_x(B)\geqslant_{\mathbb{D}}\tfrac{1}{2}|\langle x,[A,B]x\rangle|_{\mathbb{D}}, \qquad (6.5)$$

$$[A,B]=AB-BA.$$

Proof. Set $u=(A-a_xI)x$ and $v=(B-b_xI)x$. Cauchy–Schwarz gives $\Delta_x(A)\Delta_x(B)\geqslant_{\mathbb{D}}|\langle u,v\rangle|_{\mathbb{D}}$. In each complex component, the imaginary part of $\langle u,v\rangle$ equals $(1/2\mathbf{i})\langle x,[A,B]x\rangle$, so $|\langle u,v\rangle|$ is at least the modulus of that imaginary part. Applying the complex Robertson inequality componentwise yields the displayed hyperbolic order. □

Example 6.9. Paired Pauli observables. On $H=\mathscr{B}\mathbb{C}^2$, let $A=\sigma_x\mathbf{e}_1+\sigma_z\mathbf{e}_2$ and $B=\sigma_y\mathbf{e}_1+\sigma_x\mathbf{e}_2$, where $\sigma_x,\sigma_y,\sigma_z$ are the Pauli matrices. Then $[A,B]=2\mathbf{i}\sigma_z\mathbf{e}_1+2\mathbf{i}\sigma_y\mathbf{e}_2$, and the uncertainty inequality gives two standard qubit bounds simultaneously.

Proof. Matrix multiplication is componentwise and the Pauli commutators satisfy $[\sigma_x,\sigma_y]=2\mathbf{i}\sigma_z$ and $[\sigma_z,\sigma_x]=2\mathbf{i}\sigma_y$. For a normalized $x=x_1\mathbf{e}_1+x_2\mathbf{e}_2$, Equation (6.5) is exactly the Robertson inequality for (x_1,σ_x,σ_y) and for (x_2,σ_z,σ_x). □

Definition 6.10. A bicomplex frame $\{f_n\}$ has bounds $A=A_1\mathbf{e}_1+A_2\mathbf{e}_2$ and $B=B_1\mathbf{e}_1+B_2\mathbf{e}_2$ with positive coefficients when

$$A\|x\|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}\Sigma_n|\langle f_n,x\rangle|_{\mathbb{D}}^2\preccurlyeq_{\mathbb{D}}B\|x\|_{\mathbb{D}}^2. \qquad (6.6)$$

Proposition 6.11. The frame operator $Sx=\Sigma_n f_n\langle f_n,x\rangle$ is bounded, positive, self-adjoint, and invertible exactly when both lower frame-bound coefficients are positive. Reconstruction is $x=\Sigma_n S^{-1}f_n\langle f_n,x\rangle$.

Proof. The frame inequalities are the two complex frame inequalities. Their frame operators S_j have $A_jI\preccurlyeq S_j\preccurlyeq B_jI$ and are positive invertible when $A_j>0$. Recombine $S=S_1\mathbf{e}_1+S_2\mathbf{e}_2$ and the two reconstruction formulas. If one lower coefficient is zero, the corresponding component cannot be inverted and full reconstruction fails. □

Proposition 6.12. Orthogonal direct sums preserve Hilbert structure. If M is closed, the map $M\times M^\perp\to H$, $(m,n)\mapsto m+n$, is a unitary bicomplex isomorphism when the product inner product is used.

Proof. The projection theorem gives bijectivity. Orthogonality gives $\langle m+n,m'+n'\rangle=\langle m,m'\rangle+\langle n,n'\rangle$, so the map preserves the inner product. □

Polarization and the parallelogram law show that Hilbert geometry is encoded in the norm despite its partial order. The compact spectral theorem explains why separate component index sets are sometimes unavoidable. The uncertainty result is mathematically exact, but its two coefficients are two uncertainty bounds; a physical model must decide how they are observed or combined.

Frames weaken the equal-dimension requirements of a free orthonormal basis and are therefore important for redundant signal representations. Nevertheless, invertible reconstruction still requires a positive lower bound in both channels. These distinctions are central when bicomplex Hilbert spaces are used for quantum, Fourier, or learning representations.

6.13 Riesz Bases and Tensorial Remarks

Definition 6.13. A bicomplex Riesz basis is the image of a bicomplex orthonormal basis under a bounded invertible bicomplex-linear operator U.

Proposition 6.14. A paired family $\{f_n=f_{n,1}\mathbf{e}_1+f_{n,2}\mathbf{e}_2\}$ is a bicomplex Riesz basis if and only if both component families are complex Riesz bases indexed by the same set. Its optimal lower and upper bounds are the paired component bounds.

Proof. If $f_n=Uu_n$, then $U=U_1\mathbf{e}_1+U_2\mathbf{e}_2$ is bounded invertible and $f_{n,j}=U_ju_{n,j}$, so each component is a Riesz basis. Conversely, pair the two component synthesis isomorphisms to obtain U. The coefficient inequalities are exactly the two complex Riesz-basis inequalities. □

Example 6.15. On $\mathscr{B}\mathbb{C}^2$, let $U=\mathrm{diag}(2,1)\mathbf{e}_1+\mathrm{diag}(1,3)\mathbf{e}_2$ and $f_k=Ue_k$. Then $\{f_1,f_2\}$ is a Riesz basis with component lower bounds 1 and 1 and upper bounds 4 and 9 for the squared coefficient inequality.

Proof. The singular values of U_1 are 2 and 1; those of U_2 are 1 and 3. Squared lower and upper singular values give the Riesz bounds. Both component matrices are invertible, so U is bicomplex invertible. □

Proposition 6.16. If H and K have matched component Hilbert dimensions and bicomplex orthonormal bases $\{u_r\}$ and $\{v_s\}$, then the elementary tensors $u_r \otimes v_s$ form an orthonormal basis of the componentwise Hilbert tensor product

$$H\otimes K=(H_1\otimes_2 K_1)\mathbf{e}_1\oplus(H_2\otimes_2 K_2)\mathbf{e}_2. \qquad (6.7)$$

Proof. Mixed idempotent tensors vanish, and each equal-component Hilbert tensor product has the standard tensor basis. Inner products multiply componentwise, giving orthonormality, and density follows from density of finite tensor sums in each component. □

Proposition 6.17. For bounded operators A on H and B on K, the tensor operator $A\otimes B$ has

$$\|A\otimes B\|_{\mathbb{D}}=\|A_1\|\|B_1\|\mathbf{e}_1+\|A_2\|\|B_2\|\mathbf{e}_2. \qquad (6.8)$$

Proof. The Hilbert tensor-product norm of a complex tensor operator is the product of the operator norms. Apply this to both idempotent components. □

Riesz bases are more stable than arbitrary complete systems because coefficient maps are boundedly invertible. In a bicomplex setting the stability constants are two pairs, and reconstruction fails globally if either component synthesis map loses invertibility. This is relevant to numerical signal representations: conditioning can differ sharply between channels even when a scalarized condition number appears moderate.

Tensor products underlie composite quantum systems and multichannel kernels. The componentwise Hilbert tensor product is mathematically natural, but a physical theory must specify how bicomplex positivity, entanglement, and measurement interact with the idempotent split. Recent bicomplex tensor and Choi work addresses part of this problem by defining tensorial positivity for linear maps. The Hilbert-space formulas above supply norm and basis infrastructure, not a complete physical interpretation.

Theorem 6.18. Best finite-dimensional approximation. Let M be a finite-dimensional bicomplex submodule with matched component bases, and let $\{u_1,\dots,u_m\}$ be an orthonormal basis. Then $P_M x=\Sigma_{r=1}^m u_r\langle u_r,x\rangle$ and

$$\|x-P_M x\|_{\mathbb{D}}=\min_{y\in M}^{\preceq \mathbb{D}} \|x-y\|_{\mathbb{D}}. \qquad (6.9)$$

Proof. The finite expansion is the component orthogonal projection formula. For any $y\in M$, $x-y=(x-P_M x)+(P_M x-y)$ with orthogonal terms, so the squared norm is the sum of the two squared norms. Hence the projection error is no larger in either component and is uniquely minimal. □

7. Function-Space Examples and Applications

Example 7.1. Let (Ω,μ) be a measure space and $H=L^2(\Omega,\mathscr{B}\mathbb{C})=\mathbf{e}_1L^2(\Omega,\mathbb{C})\oplus\mathbf{e}_2L^2(\Omega,\mathbb{C})$. Define

$$\langle f,g\rangle=\left(\int_\Omega \bar{f}_1 g_1\, d\mu\right)\mathbf{e}_1+\left(\int_\Omega \bar{f}_2 g_2\, d\mu\right)\mathbf{e}_2. \qquad (7.1)$$

Proof. The two coefficients are complex L^2 inner products. Cauchy–Schwarz, completeness, projections onto closed submodules, and Fourier expansions follow from the component L^2 theory. □

Example 7.2. A bicomplex reproducing-kernel Hilbert space has a kernel $K(z,w)=K_1(z,w)\mathbf{e}_1+K_2(z,w)\mathbf{e}_2$, where K_j are positive complex kernels. Evaluation satisfies $f(w)=\langle K(\cdot,w),f\rangle$ and

$$|f(w)|_{\mathbb{D}}\preceq_{\mathbb{D}}\sqrt{K(w,w)}\, \|f\|_{\mathbb{D}}. \qquad (7.2)$$

Proof. The reproducing identity and evaluation inequality are the two component RKHS identities and Cauchy–Schwarz estimates. Positivity of K means positivity of both Gram matrices. □

In bicomplex quantum mechanics, a normalised free state necessitates $\langle\psi,\psi\rangle=1$, which implies that both component states have a unit norm. The expectation $\langle\psi,A\psi\rangle$ is contained in $\mathbb{D}$, and a self-adjoint operator has component observables. The Hilbert axioms are insufficient to determine a physical rule for converting the ordered pair into a single experimental probability or expectation (Rochon & Tremblay, 2004, 2006).

Fourier-type representations on $L^2(\Omega,\mathscr{B}\mathbb{C})$ are complex Fourier transforms that are paired. Componentwise, plancherel and inversion are held. This is beneficial for the representation of two complex channels; however, interaction must be facilitated by constraints, nonlinearities, or a coupled measurement. In the field of signal processing, bicomplex frames offer a stable reconstruction with two frame-bound pairings (Elgourari et al., 2021). Reconstruction fails in the component if either lower bound coefficient is zero.

Analytic interpolation and learning formulations are facilitated by reproducing kernels. Concrete Hilbert or Banach function spaces are provided by recent bicomplex Bergman, Bloch, Wiener, and Herglotz studies (Reséndis & Tovar, 2020; Alpay et al., 2023). The component theorem machinery is precise; the value that is specific to an application is contingent upon whether the paired representation captures meaningful structure.

8. Open Problems and Limitations

Equal component Hilbert dimensions are required for a free bicomplex orthonormal basis. As a result of the necessity for distinct idempotent bases due to unequal dimensions, some of the more common matrix notation becomes misleading. Domain, closure, and deficiency-index theory are necessary for both components of unbounded self-adjoint operators. Direct analogues of compact spectral sets are complicated by full spectral cylinders.

The following are open directions: spectral measures on bicomplex domains, uncertainty relations, contrived Hilbert modules, numerical algorithms that avoid null-cone division, frame theory with component imbalance, tensor products compatible with positivity, and bicomplex C*-modules and von Neumann structures. Explicit rules for states, observables, coupling, and scalar measurement outcomes are necessary in quantum and signal applications. In addition to Hilbert geometry, statistical generalisation theory is necessary for reproducing-kernel applications.

9. In summary

Bicomplex Hilbert geometry is a precise pair of complex Hilbert geometries that are connected by idempotent algebra. Even when a nonzero value is a zero divisor, the hyperbolic norm is positive and definite. Component proofs establish the following: Cauchy–Schwarz, triangle and parallelogram identities, Gram–Schmidt under free-independence hypotheses, Bessel and Parseval formulas, orthogonal projections, and Riesz representation. The decomposition of adjoint and spectral operator classes is also exact. The framework is capable of supporting L^2, Fourier, frame, reproducing-kernel, signal, and quantum formulations; however, it does not independently provide cross-channel physics or learning performance. Unbounded operators, null-cone normalisation, unequal component dimensions, and positivity-compatible tensor products continue to be critical research issues.

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