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Applications of Bessel Polynomials in Time Domain Synthesis Problem

P V Maya[1], T M Vidya[2], P C Sreenivas[3] and T M Vasudevan Nambisan[4]

[1]Assistant Professor, Department of mathematics, Mahatma Gandhi College, Iritty, Kannur university, Kerala
[2]Assistant Professor, Department of mathematics, Mahatma Gandhi College, Iritty Kannur University, Kerala, India
[3]Retired Principal, Payyanur College, Payyanur, Kannur University, Kerala, India
[4]Emerits Professor, College of Engineering, Trikaripur, India,

Abstract

The object of this paper is to evaluate an integral involving Bessel polynomials and I- function of several variables and to apply it on obtaining a particular solution of the classical problem known as the "Time domain synthesis problem", occurring in the electrical network theory. Special cases include a result given by Yaswant Singh and Athul Garg [18].

Key Words: Multivariable I –functions , Bessel Polynomials

1. Introduction

This paper deals with the evaluation of an integral involving product of Bessel polynomials and $\bar{I}$ function of *r*- variables. By making use of this integral the solution of the time-domain synthesis problem is investigated.

Notations and Results used :

$(a)_n$ stands for $a(a+1)...(a+n-1)$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, n \geq 1$$

${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),...,(a_p;\alpha_p,A_p)$

$$n = \frac{\Gamma(n+1)}{\Gamma n}, n \geq 1$$

Prathima and Vasudevan Nambisan [8, p.38] defined *I*-function of *r*-variables and is represented as,

$$I[z_1,.....,z_r]$$

$$= I^{0,N:\ m_1,n_1;\ldots;m_r,n_r}_{P,Q:\ p_1,q_1;\ldots;p_r,q_r}\left[\begin{array}{l|l} z_1 & {}_1(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)};A_j)_P:{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ldots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)};B_j)_Q:{}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ldots;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\ \omega\ )^r}\int_{L_1}\ldots\int_{Lr}\theta_1(s_1)\ldots\theta_r(s_r)\ \phi\ (s_1,\ldots,s_r)\ z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r, \qquad (1.1)$$

where, $\phi\ (s_1,\ldots,s_r)$ and $\theta_i(s_i),\ i=1,2,\ldots,r$ are given by,

$$\phi\ (s_1,\ldots,s_r)=\frac{\prod_{j=1}^{N}\Gamma^{A_j}(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{Q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\prod_{j=N+1}^{P}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)}, \qquad (1.2)$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}, \qquad (1.3)$$

Also $z_i\neq 0\ \ (i=1,\ldots,r),\ \ \omega=\sqrt{-1},\ \ m_j,n_j,p_j,q_j\ \ (j=1,\ldots,r)$, N , P , Q are non-negative integers such that $0\le N\le P,\ \ Q\ge 0,\ \ 0\le m_j\le q_j,\ \ 0\le n_j\le p_j\ \ (j=1,2\ldots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}\left(j=1,2,\ldots,P\ ,i=1,2\ldots,r\right),\ \beta_j^{(i)}\left(j=1,2,\ldots,Q,i=1,2,\ldots,r\right),\ \gamma_j^{(i)}\left(j=1,2,\ldots,p_i,i=1,2,\ldots,r\right)$, and $\delta_j^{(i)}\left(j=1,2,\ldots,q_i,i=1,2,\ldots,r\right)$ are positive numbers. $a_j\left(j=1,2,\ldots,P\right),\ \ b_j\left(j=1,2,\ldots,Q\right),$ $c_j^{(i)}\left(j=1,2,\ldots,p_i,i=1,2,\ldots,r\right)$, and $d_j^{(i)}\left(j=1,2,\ldots,q_i,i=1,2,\ldots,r\right)$ are complex numbers. The exponents $A_j\left(j=1,2,\ldots,P\right),\ B_j\left(j=1,2,\ldots,Q\right),\ C_j^{(i)}\left(j=1,2,\ldots,p_i,i=1,2,\ldots,r\right)$, and $D_j^{(i)}\left(j=1,2,\ldots,q_i,i=1,2,\ldots,r\right)$ of various gamma functions may take non integer values.

The *I*-function of *r*-variables is analytic if

$$\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\ \le 0, i=1,2,\ldots,r.$$

The integral (1.1) converges absolutely if $\left|\ \arg(z_i)\ \right|<\frac{1}{2}\Delta_i\pi\ \ ,\ \ i=1,2,\ldots,r$,

where, $\Delta_i = \left(- \sum_{j=n+1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} \delta_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \gamma_j^{(i)} \right) > 0$ (1.4) On

taking $D_j^{(i)} = 1 \left(j = 1,2,...,m_i, i = 1,2,...,r \right)$ in (1.1), then I- function will be denoted by

$$\bar{I}[z_1,.....,z_r] = I_{P,Q:\ p_1,q_1;...;p_r,q_r}^{0,N:\ m_1,n_1;...;m_r,n_r} \begin{bmatrix} z_1 & I_1 \\ \vdots & \\ z_r & I_2 \end{bmatrix}$$

$$= \frac{1}{(2\pi\ \omega\ )^r} \int_{L_1} ... \int_{Lr} \bar{\theta}_1(s_1) ... \bar{\theta}_r(s_r)\ \phi\ (s_1,...,s_r)\ z_1^{s_1} ... z_r^{s_r}\, ds_1 ... ds_r, \qquad (1.5)$$

Where,

$$I_1 = {}_1(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};...;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}$$

$$I_2 = {}_1(b_j;\beta_j^{(1)},...,\beta_j^{(r)};B_j)_Q\ :\ {}_1\left(d_j{}^{(1)},\delta_j{}^{(1)};1\right)m_1;\ {}_{m_1+1}(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};$$

$$...;\ \ {}_1\left(d_j{}^{r},\delta_j{}^{r};1\right)_{m_r};\ {}_{m_r+1}(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \qquad (1.6)$$

$$\bar{\theta}_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma\ (d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}} (1 - c_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}} (1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}} (c_j^{(i)} - \gamma_j^{(i)} s_i)},$$

where , $i = 1,2,...r.$

The integral (1.5) converges absolutely if $\left| \arg(z_i) \right| < \frac{1}{2} \Delta'_i\, \pi, i = 1,2,...,r.$

where,

$$\Delta'_i = \left(- \sum_{j=n+1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} \delta_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \gamma_j^{(i)} \right) > 0,\ i = 1,2,...,r. \qquad (1.7)$$

Orthogonality property of Bessel polynomials is derived by (Exton 4, p.215) is

$$\int_0^{\infty} x^{a-2} e^{-\frac{x}{2}} y_m(x;a,1)\, y_n(x;a,1)\, dx = \frac{(-1)^m\, n!(n+a-1)\pi}{\Gamma(a+n)(2n+a-1)\sin \pi a} \delta_{mn} \qquad (1.8)$$

where

$$\mathrm{Re}(a) < 1 - m - n$$

The integral defined by Bajpai [1] is

$$\int_0^\infty x^{\sigma-1} e^{-\frac{x}{2}} y_n(x;a,1)\,dx = \frac{\Gamma(-\sigma-n)\Gamma(a-\sigma-1+n)}{\Gamma(a-n-1)} \tag{1.9}$$

where

$$\mathrm{Re}(-\sigma+n) < 0, \mathrm{Re}(a-\sigma+n-1) > 0, \quad \sigma \neq -1,-2,-3,\ldots$$

2. Required Integral

$$\int_0^\infty x^{\sigma-1} e^{-\frac{x}{2}} y_n(x;a,1)\,\bar{I}\left[z_1 x^\lambda, z_2, \ldots, z_r\right]dx = I^{0,0:\ m_1+2,n_1;.m_2,n_2..;m_r,n_r}_{P,Q:\ p_1+1,q_1+2;p_2,q_2;\ldots;p_r,q_r} \begin{bmatrix} z_1 x^\lambda & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \\ & \end{bmatrix} \tag{2.1}$$

$$I_1 = {}_1\left(a_j, \alpha_j^{(1)}, \ldots \alpha_j^{(r)}; A_j\right)_P : {}_1(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{p_1}, (a-\sigma-1, \lambda; 1); {}_1(c_j^{(2)}, \gamma_j^{(2)}; C_j^{(2)})_{p_2}; \ldots; {}_1(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{p_r} \tag{2.2}$$

$$I_2 = (-\sigma-n, \lambda; 1), (a-\sigma+1+n, \lambda; 1), {}_1\left(b_j, \beta_j^{(1)}, \ldots, \beta_j^{(r)}; B_j\right)_Q ; {}_1(d_j^{(1)}, \delta_j^{(1)}; 1)_{m_1}; {}_{m_1+1}(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{q_1};$$
$$\ldots ; {}_1\left(d_j^{\,r}, \delta_j^{\,r}; 1\right)_{m_r} ; {}_{m_r+1}(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{q_r}$$

where,

$$\mathrm{Re}\left(-\sigma-n-\lambda\frac{(c_j^{(1)}-1)}{\gamma_j}\right) > 0, \quad \mathrm{Re}\left(a-\sigma+n+1-\lambda\frac{(d_j^{(1)}-1)}{\delta_j}\right) > 0$$
$$j = 1,2,\ldots m, \quad \sigma \neq -1,-2,-3,\ldots$$

(2.3)

(2.4)

and $\left|\ \arg(z_i)\ \right| < \frac{1}{2}\Delta'_i\,\pi, i = 1,2,\ldots,r$ (2.5)

where

$\Delta'_i, i = 1, 2, ..., r$ is given by (1.4)

Proof:

To prove (2.1), express the I- function of several variables as a Melin Barnes contour integral given by (1.1). Interchange the order of integration(which is permitted under the condition stated) and evaluate the inner integral with the help of (1.9). Interpretting the integral with the help of (1.5) to get the required integral.

3. Solution of the Time-Domain Synthesis Problem of Signals

The classical Time- domain synthesis problem occurring in electrical network theory as follows [5,p.139].

Given an electric signal dissolved by a real valued conventional function $f(t)$ on $0 < t < \infty$ construct an electric network consisting of a finite number components *R*,*C* and *I* which are all fixed linear and positive such that the output of $f(t)$ resulting from a delta function $\delta(t)$ approximates $f(t)$ in $0 < t < \infty$ in some sense.

In order to obtain a solution of this problem, we expand the function $f(t)$ in to a convergent series.

$$f(t) = \sum_{n=0}^{\infty} \psi_n(t) \qquad (3.1)$$

for a real valued function $\psi_n(t)$.

Let every partial sum

$$f_N(t) = \sum_{n=0}^{N} \psi_n(t), \quad N = 0, 1, 2, ... \qquad (3.2)$$

possess the two properties.

1. $f_N(t) = 0 \quad for \quad -\infty < t < 0$

2. The Laplace transform $f_N(s)$ *of* $f_N(t)$ is a rational function having a zero as $s = \infty$ and all its poles in the left hand plane ,except possibly for a simple pole at the origin.

After choosing N in (3.2) sufficiently large to satisfy whatever approximation criterion is being used, an orthonormal series expansion may be employed. The Bessel polynomial transformation and (1.1) yield an immediate solution in the following form.

$$f(t) = \sum_{n=0}^{\infty} C_n t^{\left(\frac{a-2}{2}\right)} e^{-\frac{t}{2}} y_n(t; a, 1)$$

where

$$C_n = \frac{(-1)^n \Gamma(a+n)(2n+a-1)\sin \pi a}{n!(n+a-1)\pi} \int_0^\infty f(t) t^{a-2} y_n(t;a,1)\, dt \qquad (3.3)$$

where

$$\mathrm{Re}(a) < 1-2n$$

The function $f(t)$ is continuous and bounded variation in the open interval $(0,\infty)$.

4. Particular Solution of the problem

$$f(t) = \frac{\sin \pi a}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(a+n)(2n+a-1)}{n!(n+a-1)} t^{\left(\frac{a-2}{2}\right)} e^{-\frac{t}{2}}$$

$$\times I_{P,Q:\ p_1+1,q_1+2;p_2,q_2;\ldots;p_r,q_r}^{0,0:\ m_1+2,n_1;.m_2,n_2..;m_r,n_r} \left[\begin{array}{c|c} z_1 & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \\ & \end{array}\right] y_n(t;a,1)$$

where I_1 and I_2 are given by (2.2) and (2.3) respectively.

provided by the following conditions.

$$i) \qquad \sigma < 0, \quad \mathrm{Re}(a) < 1-2n$$

$$ii) \ \mathrm{Re}\left(a-\sigma+\frac{(a_j-1)}{\alpha_j}\right) < 2, \quad j=1,2,\ldots m_i, \quad i=1,2,\ldots,r, \quad \sigma \neq -1,-2,-3,\ldots$$

and result (2.1) also holds.

Proof:

Let us consider

$$f(t) = t^{\left(\sigma-\frac{a}{2}\right)} e^{-\frac{t}{2}} \overline{I}\left[z_1 x^{\lambda}, z_2, \ldots, z_r\right]$$

$$= \sum_{m=0}^{\infty} C_m t^{\left(\frac{a-2}{2}\right)} e^{-\frac{t}{2}} y_m(t;a,1) \qquad (4.2)$$

Equation (4.2) is valid, since $f(t)$

is continuous and bounded variation in the open interval $(0,\infty)$.

Multiplying both sides of (4.2) by $t^{\left(\frac{a-2}{2}\right)} e^{-\frac{t}{2}} y_n(t;a,1)$

and integrating with respect to t from 0 to ∞,to get,

$$\int_0^{\infty} t^{(\sigma-1)} e^{-\frac{t}{2}} y_n(t;a,1)\, \bar{I}\left[z_1 x^{\lambda}, z_2, \ldots, z_r\right] dt$$

$$= \sum_{m=0}^{\infty} C_m \int_0^{\infty} t^{(a-2)} e^{-\frac{t}{2}} y_m(t;a,1)\, y_n(t;a,1)\, dt \qquad (4.3)$$

Now using (2.1), the left hand side f the expression in (4.3) becomes,

$$\int_0^{\infty} t^{(\sigma-1)} e^{-\frac{t}{2}} y_n(t;a,1)\, \bar{I}_1\left[z_1 x^{\lambda}, z_2, \ldots, z_r\right] dt$$

$$= I^{0,0:\ m_1+2, n_1; m_2, n_2 \ldots; m_r, n_r}_{P,Q:\ p_1+1, q_1+2; p_2, q_2; \ldots; p_r, q_r} \left[\begin{array}{c|c} z_1 & I_1 \\ z_2 & \\ \vdots & \\ z_r & I_2 \\ & \end{array}\right] \qquad (4.4)$$

where I_1 and I_2 are given by (2.2) and (2.3) respectively.

Now by using (1.8) , the expression (4.3) reduces to:

$$\int_0^{\infty} t^{(\sigma-1)} e^{-\frac{t}{2}} y_n(t;a,1)\, \bar{I}\left[z_1 x^{\lambda}, z_2, \ldots, z_r\right] dt$$

$$= C_m (-1)^n \frac{n!(n+a-1)\pi}{\Gamma(a+n)\Gamma(2n+a-1)\sin \pi a} \qquad (4.5)$$

Combining (4.4) and (4.5),to get,

$$C_n = (-1)^n \frac{\Gamma(a+n)\Gamma(2n+a-1)\sin \pi a}{n!(n+a-1)\pi} \bar{I}^{0,0:\ m_1+2,n_1;.m_2,n_2..;m_r,n_r}_{P,Q:\ p_1+1,q_1+2;p_2,q_2;...;p_r,q_r} \left[\begin{array}{c|c} z_1 & I_1 \\ \vdots & \\ z_r & I_2 \end{array} \right]$$ where

I_1 and I_2 are given by (2.2) and (2.3) respectively.

Substituting the value of C_m in (4.2), to get the required result.

Putting P = Q = 0 and r = 1 to get the result given by Yanswath Singh and Athul Garg.

On account of the most general character of the result (4.2) due to the presence of I- functions of several variables, many known and unknown results can be derived.

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